Solutions Manual-Theory of Vibration with Applications
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1-1

$x = A \sin \omega t$ $\qquad A = 0.20$ cm $\qquad \tau = 0.15$ s

$\omega = \frac{2\pi}{\tau} = 41.89$ r/s

$\dot{x} = \omega A \cos \omega t$ $\qquad \dot{x}_{max} = \omega A = 8.38$ cm/s

$\ddot{x} = -\omega^2 A \sin \omega t$ $\qquad \ddot{x}_{max} = \omega^2 A = 350.9$ cm/s²

1-2

$\omega = 2\pi f = 2\pi \times 82 = 515.2$ r/s $\qquad \omega^2 = 0.2655 \times 10^6$

$g = 980.4$ cm/s²

$$x_{max} = \ddot{x}_{max}/\omega^2 = \frac{50 \times 980.4}{0.2655 \times 10^6} = 0.184 \text{ cm}$$

1-3

$\omega = 2\pi f = 2\pi \times 10 = 62.83$ r/s $\qquad \tau = \frac{1}{f} = 0.10$ s

$\dot{x}_{max} = \omega A = 4.57$ m/s $\qquad A = 0.07274$ m $= 7.274$ cm

$\ddot{x}_{max} = \omega^2 A = 287.1$ m/s²

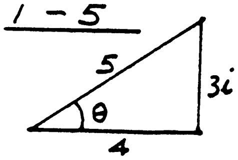
1-4

$$x = A(\sin \omega_1 t + \sin \omega_2 t) = 2A \cos \tfrac{1}{2}(\omega_1 - \omega_2)t \cdot \sin \tfrac{1}{2}(\omega_1 + \omega_2)t$$

let $\omega_1 = \omega$, $\quad \omega_2 = \omega + \Delta\omega$, $\quad \omega_1 + \omega_2 = 2\omega + \Delta\omega \cong 2\omega$

$\therefore\ x \cong 2A \cos \tfrac{1}{2} \Delta\omega t \cdot \sin \omega t$

1-5

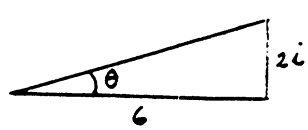


$z = 4 + 3i = 5(\cos\theta + i \sin\theta) = 5e^{i\theta}$

$\theta = \tan^{-1}\frac{3}{4} = 36°52' = 0.6435$ r

1-6

$$\begin{array}{r} 2 + 3i \\ 4 - i \\ \hline \text{sum} = 6 + 2i \end{array}$$



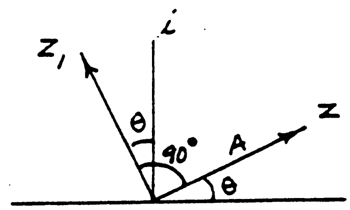
$z = Ae^{i\theta}$ $\qquad A = \sqrt{6^2 + 2^2} = 6.325$

$\theta = \tan^{-1}\frac{2}{6} = 18°26' = 0.3217$ r

$z = 6.325\, e^{0.3217 i}$

$= 6.325 \underline{/18°26'}$

1-7

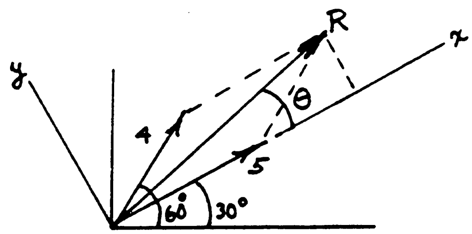


$$z = A(\cos\theta + i\sin\theta) = Ae^{i\theta}$$

$$iz = A(i\cos\theta - \sin\theta) = z_1$$

$$= A[\cos(\theta+90) + i\sin(\theta+90)]$$

1-8



$$\frac{\pi}{6} = 30°$$

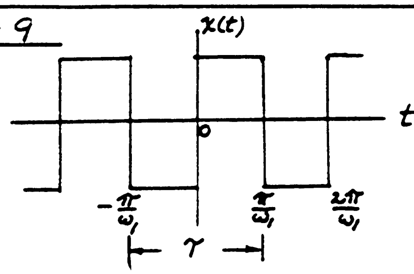
$$R_x = 5 + 4\cos 30° = 5 + 3.47$$
$$= 8.47$$

$$R_y = 4\sin 30° = 2.00$$

$$R = \sqrt{8.47^2 + 2.0^2} = 8.70$$

$$\theta = \tan^{-1}\frac{2}{8.70} = 12°57'$$

1-9



$x(t)$ is odd function $\therefore a_n = 0$

$$b_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} x(t)\sin n\omega_1 t\, dt$$

$$\tau = \frac{2\pi}{\omega_1}$$

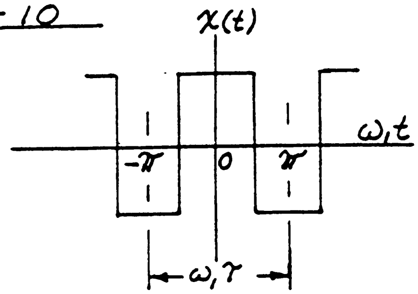
$$b_n = \frac{\omega_1}{\pi}\left[\int_{-\frac{\pi}{\omega_1}}^{0}(-1)\sin n\omega_1 t\, dt + \int_{0}^{\frac{\pi}{\omega_1}}(+1)\sin n\omega_1 t\, dt\right]$$

$$= \frac{\omega_1}{\pi}\left[\frac{\cos n\omega_1 t}{n\omega_1}\Bigg|_{-\frac{\pi}{\omega_1}}^{0} - \frac{\cos n\omega_1 t}{n\omega_1}\Bigg|_{0}^{\frac{\pi}{\omega_1}}\right] = \frac{2}{n\pi}(1 - \cos n\pi)$$

$$\therefore b_n = \begin{cases} 0 & \text{for } n \text{ even} \\ \frac{4}{n\pi} & \text{"} \ n \text{ odd} \end{cases}$$

$$x(t) = \frac{4}{\pi}\left(\sin\omega_1 t + \frac{1}{3}\sin 3\omega_1 t + \frac{1}{5}\sin 5\omega_1 t + \cdots\right)$$

1 - 10



$x(t)$ is even function

$b_m = 0$

$$a_m = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} x(t)\cos m\omega_1 t \, dt$$

$$\tau = \frac{2\pi}{\omega_1} \qquad \omega_m = m\omega_1$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{-\pi/2}(-1)\cos n\omega_1 t \, d(\omega_1 t) + \frac{1}{\pi}\int_{-\pi/2}^{+\pi/2}(+1)\cos m\omega_1 t \, d(\omega_1 t) + \frac{1}{\pi}\int_{\pi/2}^{\pi}(-1)\cos n\omega_1 t \, d(\omega_1 t)$$

$$= \frac{1}{m\pi}\left\{\pm 4\right\} \quad \begin{matrix} + \text{ for } n = 1, 5, 9, \cdots \\ - \text{ for } m = 3, 7, 11, \cdots \end{matrix}$$

$$x(t) = \frac{4}{\pi}\left(\cos\omega_1 t - \frac{1}{3}\cos 3\omega_1 t + \frac{1}{5}\cos 5\omega_1 t \cdots\right)$$

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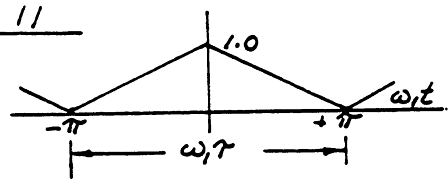
1 - 11



$x(t)$ is even function $b_m = 0$

$$x(t) = \begin{cases} \frac{1}{\pi}(t+\pi) & -\pi \le t \le 0 \\ \frac{1}{\pi}(\pi - t) & 0 \le t \le \pi \end{cases}$$

$$\frac{1}{2}a_0 = \text{average value} = \frac{1}{2}$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{0}\frac{1}{\pi}(t+\pi)\cos n\omega_1 t \, d(\omega_1 t) + \frac{1}{\pi}\int_{0}^{\pi}\frac{1}{\pi}(\pi - t)\cos m\omega_1 t \, d(\omega_1 t)$$

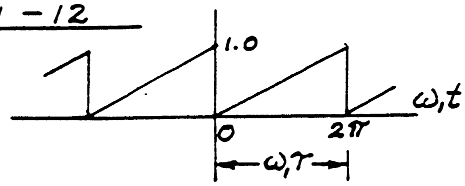
$$= \frac{2}{\pi}\int_0^{\pi}\frac{1}{\pi}(\pi - t)\cos n\omega_1 t \, d(\omega_1 t) \qquad \text{because of symmetry}$$

$$= \frac{2}{\pi}\left.\frac{\sin n\omega_1 t}{n}\right|_0^{\omega_1 t = \pi} - \frac{2}{\pi^2}\left.\left(\frac{\cos m\omega_1 t}{m^2} + \omega_1 t\,\frac{\sin n\omega_1 t}{n}\right)\right|_0^{\omega_1 t = \pi}$$

$$= \begin{cases} 0 & \text{for } n \text{ even} \\ \frac{4}{m^2\pi^2} & \text{for } n \text{ odd} \end{cases}$$

$$x(t) = \frac{1}{2} + \frac{4}{\pi^2}\left(\cos\omega_1 t + \frac{1}{3^2}\cos 3\omega_1 t + \frac{1}{5^2}\cos 5\omega_1 t + \cdots\right)$$

1 - 12



$$x = \frac{\omega_1 t}{2\pi} \qquad 0 \le \omega_1 t \le 2\pi$$

$$c_n = \frac{\omega_1}{2\pi}\int_0^{2\pi} \frac{\omega_1 t}{2\pi} e^{-in\omega_1 t}\, dt$$

$$c_n = \frac{1}{(2\pi)^2}\int_0^{2\pi} \omega_1 t\, e^{-in\omega_1 t}\, d(\omega_1 t) = \frac{1}{(2\pi)^2}\left[\frac{e^{-in\theta}}{(-in)^2}\left(-in\theta - 1\right)\right]_0^{2\pi} \quad \text{where } \theta = \omega_1 t$$

$$= \frac{1}{(2\pi)^2 n^2}\left[-1 + (1 + i2\pi n)\, e^{-i2\pi n}\right] = \frac{i}{2\pi}\,\frac{1}{n}$$

$$x(t) = c_0 + \frac{i}{2\pi}\left[(e^{i\omega_1 t} - e^{-i\omega_1 t}) + \frac{1}{2}(e^{i2\omega_1 t} - e^{-i2\omega_1 t}) + \frac{1}{3}(e^{i3\omega_1 t} - e^{-i3\omega_1 t}) + \cdots\right]$$

$$= \frac{1}{2} - \frac{1}{\pi}\left[\sin \omega_1 t + \frac{1}{2}\sin 2\omega_1 t + \frac{1}{3}\sin 3\omega_1 t + \cdots\right]$$

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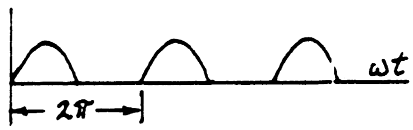
1 - 13



$$\overline{x^2} = \lim_{T\to\infty} \frac{1}{T}\int_0^{T/2} A^2 \sin^2\omega t\, dt = \lim_{T\to\infty} \frac{A^2}{T}\int_0^{T/2} \frac{1}{2}(1 - \cos 2\omega t)\, dt$$

$$= \lim_{T\to\infty} \frac{A^2}{2}\left(\frac{t}{2} - \frac{\sin 2\omega t}{4\omega}\right)\Bigg|_0^{T/2} = \lim_{T\to\infty}\left(\frac{A^2}{4} - \frac{\sin\omega T}{\omega T}\right) = \frac{A^2}{4}$$

$$\therefore\ x_{RMS} = \sqrt{\overline{x^2}} = \frac{A}{2}$$

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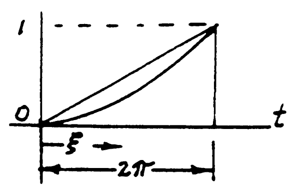
1 - 14

$$x(t) = \frac{\xi}{2\pi} \qquad x^2(t) = \frac{1}{4\pi^2}\xi^2 \qquad 0 \le \xi \le 2\pi$$



$$\overline{x^2} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{4\pi^2}\xi^2\right) d\xi = \frac{1}{2\pi}\left(\frac{1}{4\pi^2}\frac{\xi^3}{3}\right)\Bigg|_0^{2\pi}$$

$$= \frac{1}{2\pi}\left(\frac{2\pi}{3}\right) = \frac{1}{3}$$

1-14 cont.

F.S. of saw tooth wave =

$$x(t) = \frac{1}{2} - \frac{1}{\pi}\left(\sin\omega_1 t + \frac{1}{2}\sin 2\omega_1 t + \frac{1}{3}\sin 3\omega_1 t + \cdots\right)$$

$$x^2(t) = \frac{1}{4} - \frac{1}{\pi}\left(\sin\omega_1 t + \frac{1}{2}\sin 2\omega_1 t + \cdots\right) + \frac{1}{\pi^2}\left(\sin\omega_1 t + \frac{1}{2}\sin 2\omega_1 t + \cdots\right)^2$$

$$= \frac{1}{4} - \frac{1}{\pi}\left(\sin\omega_1 t + \frac{1}{2}\sin 2\omega_1 t + \cdots\right) + \frac{1}{\pi^2}\left(\sin^2\omega_1 t + \sin^2 2\omega_1 t + \cdots\right) + \text{cross products which will integrate to zero}$$

$$\frac{1}{T}\int_0^T x^2(t)\,dt = \frac{1}{4} + \frac{1}{\pi\omega_1 T}\cos\omega_1 t\Big|_0^T + \frac{1}{4\pi\omega_1 T}\cos 2\omega_1 t\Big|_0^T + \frac{1}{9\pi\omega_1 T}\cos 3\omega_1 t\Big|_0^T$$

$$+ \frac{1}{\pi^2 T}\,\frac{1}{2}\left(t - \frac{\sin 2\omega_1 t}{2\omega_1}\right)\Big|_0^T + \frac{1}{4\pi^2 T}\,\frac{1}{2}\left(t - \frac{\sin 4\omega_1 t}{4\omega_1}\right)\Big|_0^T + \cdots$$

Let $\omega_1 T = 2\pi k$ where k is an integer $\to \infty$

$$\overline{x^2} = \lim_{k\to\infty} \frac{1}{2\pi k}\int_0^{2\pi k} x^2(t)\,dt = \frac{1}{4} + \frac{1}{2\pi^2} + \frac{1}{2}\frac{1}{(2\pi)^2} + \frac{1}{2}\frac{1}{(3\pi)^2} + \cdots$$

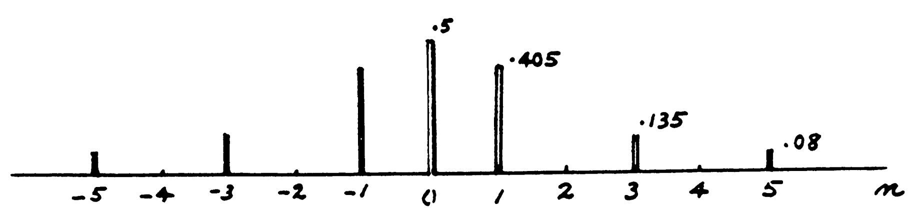
$$= \frac{1}{4} + \frac{1}{2\pi^2}\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\right) = \frac{1}{3}$$

1-15

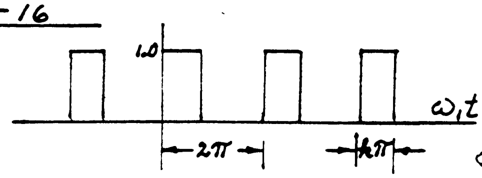
$$x(t) = \frac{1}{2} + \frac{4}{\pi^2}\left(\cos\omega_1 t + \frac{1}{3}\cos 3\omega_1 t + \frac{1}{5}\cos 5\omega_1 t + \cdots\right)$$

Fourier Spectrum = plot of coefficients. For this case $b_n = 0$

$C_n = \sqrt{a_n^2 + b_n^2} = a_n \qquad c_0 = \frac{a_0}{2}$



1-16



$k = \frac{2}{3}$

$$c_0 = \frac{a_0}{2} = \text{average value} = \frac{k\pi}{2\pi} = \frac{k}{2} = \frac{1}{3}$$

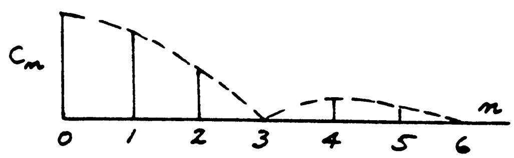
$$a_n = \frac{1}{n\pi} \sin nk\pi = \frac{1}{n\pi} \sin \frac{n}{3} 2\pi$$

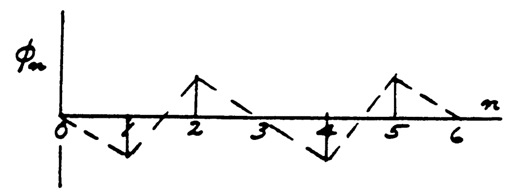
$$b_n = \frac{1}{n\pi}(1 - \cos nk\pi) = \frac{1}{n\pi}\left(1 - \cos \frac{n}{3} 2\pi\right)$$

$$2C_n = \sqrt{a_n^2 + b_n^2} = \frac{\sqrt{2}}{n\pi}\sqrt{\left(1 - \cos \frac{n}{3} 2\pi\right)} \qquad \phi_n = \tan^{-1} \frac{1 - \cos\frac{n}{3}2\pi}{\sin \frac{n}{3} 2\pi}$$

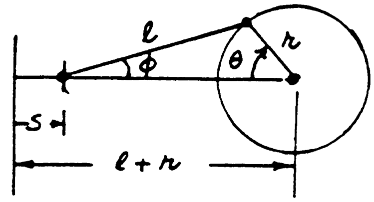
n	C_n	ϕ_n
1	.2758	−60°
2	.1379	60°
3	0	0

n	C_n	ϕ_n
4	.0689	−60°
5	.0552	60°
6	0	0





1-17



$$\ell + r - s = r\cos\theta + \ell\cos\phi$$

$$\ell \sin\phi = r\sin\theta$$

$$\therefore \cos\phi = \left[1 - \left(\frac{r}{\ell}\right)^2 \sin^2\theta\right]^{1/2}$$

$$= 1 - \frac{1}{2}\left(\frac{r}{\ell}\right)^2 \sin^2\theta - \frac{1}{8}\left(\frac{r}{\ell}\right)^4 \sin^4\theta \cdots$$

$$s = r\left[1 - \cos\theta + \frac{1}{2}\left(\frac{r}{\ell}\right)\sin^2\theta + \frac{1}{8}\left(\frac{r}{\ell}\right)^3 \sin^4\theta + \cdots\right]$$

using $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$, $\quad \sin^4\theta = \frac{1}{4}\left(\frac{3}{2} - 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right)$

$$s = r\left[1 - \cos\theta + \frac{1}{2}\left(\frac{r}{\ell}\right)\frac{1}{2}(1 - \cos 2\theta) + \cdots\right]$$

$$= r\left[1 + \frac{1}{4}\left(\frac{r}{\ell}\right) - \cos\theta - \frac{1}{4}\left(\frac{r}{\ell}\right)\cos 2\theta + \cdots\right]$$

which retains only $\left(\frac{r}{\ell}\right)$ to first power

Ratio of 2nd harmonic / 1st harmonic $= \frac{1}{4}\left(\frac{r}{\ell}\right) = \frac{1}{12}$

1-18

$$\overline{x^2} = \lim_{T\to\infty} \frac{1}{T}\int_0^T x^2(t)\,dt = \frac{A^2 k\tau}{\tau} = kA^2$$
$$= 0.10 A^2$$

$$rms = \sqrt{\overline{x^2}} = 0.3162\,A$$

1-19

$$x = \left(1 - \frac{t}{\pi}\right) \qquad 0 \le t \le \pi$$

$$x^2 = 1 - \frac{2t}{\pi} + \frac{t^2}{\pi^2}$$

$$\overline{x^2} = \frac{1}{\pi}\int_0^{\pi}\left(1 - \frac{2t}{\pi} + \frac{t^2}{\pi^2}\right)dt = \frac{1}{3}$$

1-20

$$Db = 20\log_{10}\left(\frac{x_1}{x_2}\right) = 0.50$$

$$\log_{10}\left(\frac{x_1}{x_2}\right) = \frac{0.50}{20} = 0.0250$$

$$\left(\frac{x_1}{x_2}\right) = 10^{0.0250} = 1.0593$$

$$x_1 = 1.0593\,x_2 = 1.0593 \times 2.5\,mm = 2.6481$$

$$\text{Error} = 0.0593 \times 2.5\,mm = \pm\,0.148\ mm$$

1-21

$$Db = 20\log_{10}(10) = 20$$
$$Db = 20\log_{10}(50) = 33.98$$
$$Db = 20\log_{10}(100) = 40.0$$

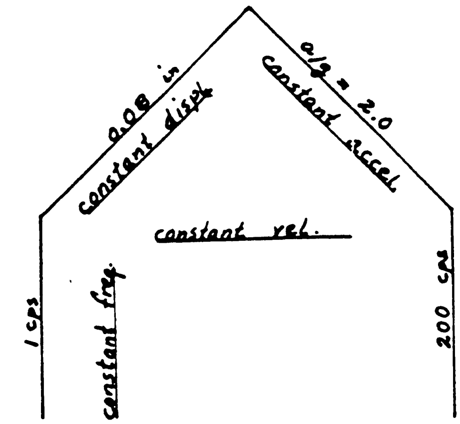
1-22

$$Db = 20\log_{10}\left(\frac{x_p}{x_{1000}}\right) = 32.$$

$$\log_{10}\left(\frac{x_p}{x_{1000}}\right) = \frac{32}{20} = 1.60$$

$$\frac{x_p}{x_{1000}} = 10^{1.60} = 39.8$$

1 - 23



1 - 24

Mean Value $= 1 * P(1) + (-1) * P(-1) = 0$

Mean square value $= (1)^2 * P(1) + (-1)^2 * P(-1) = 1$

1 - 25

$$f(t) = \frac{1}{2}\left(f(t) - f(-t)\right) + \frac{1}{2}\left(f(t) + f(-t)\right)$$

$$= O(t) + E(t)$$

$$O(-t) = \frac{1}{2}\left(f(-t) - f(t)\right) = -O(t) \quad \text{(odd function)}$$

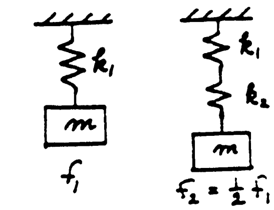
$$E(-t) = \frac{1}{2}\left(f(-t) + f(t)\right) = E(t) \quad \text{(even function)}$$

2-1

From Eq. 2.2-9

$$f = \frac{15.76}{\sqrt{\Delta_{mm}}} = \frac{15.76}{\sqrt{7.87}} = 5.62 \quad Hz$$

2-2

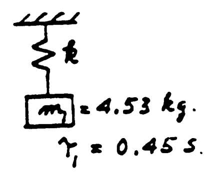


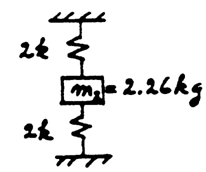
$$f_1 = \sqrt{\frac{k_1}{m}} \qquad f_2 = \sqrt{\frac{k_1 k_2}{m(k_1+k_2)}}$$

$$\frac{1}{2}\sqrt{\frac{k_1}{\cancel{m}}} = \sqrt{\frac{k_1 k_2}{\cancel{m}(k_1+k_2)}}$$

$$\frac{1}{4}\cancel{k_1} = \frac{\cancel{k_1} k_2}{k_1+k_2} \qquad \therefore \quad k_2 = \frac{1}{3} k_1$$

2-3





$$k = \left(\frac{2\pi}{\tau_1}\right)^2 m_1 = \left(\frac{2\pi}{.45}\right)^2 4.53 = 883.5 \quad N/m$$

$$\tau_2 = 2\pi\sqrt{\frac{m_2}{4k}} = 2\pi\sqrt{\frac{2.26}{4\times 883.}} = 0.159 \quad s$$

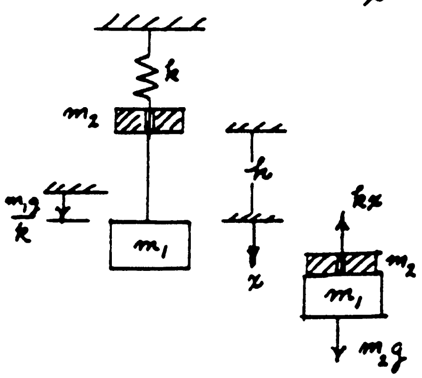
2-4

$$\frac{k}{m} = (2\pi f)^2 = \left(2\pi \frac{94}{60}\right)^2 \qquad \frac{k}{m+.453} = \left(2\pi \frac{76.7}{60}\right)^2$$

$$\frac{m+.453}{m} = \left(\frac{94}{76.7}\right)^2 \qquad \therefore \quad m = 0.9028 \text{ kg}$$

$$k = 87.48 \text{ N/m}$$

2-5



x measured from static equilibrium position of m_1 k

Eq. of motion after impact

$$(m_1+m_2)\ddot{x} = -kx + m_2 g$$

Gen solution:

$$x(t) = \frac{m_2 g}{k} + A\sin\omega t + B\cos\omega t$$

Initial conditions

$$x(0) = 0 = \frac{m_2 g}{k} + B \quad \therefore B = -\frac{m_2 g}{k}$$

$$\dot{x}(0) = \frac{m_2\sqrt{2gh}}{m_1+m_2} = \omega A \qquad \therefore \quad A = \frac{m_2\sqrt{2gh}}{(m_1+m_2)\omega}$$

$$\omega = \sqrt{\frac{k}{m_1+m_2}}$$

$$\therefore x(t) = \frac{m_2 g}{k} + \frac{m_2\sqrt{2gh}}{m_1+m_2}\sqrt{\frac{m_1+m_2}{k}}\sin\omega t - \frac{m_2 g}{k}\cos\omega t$$

$$= \frac{m_2 g}{k}(1-\cos\omega t) + \frac{m_2\sqrt{2gh}}{\sqrt{k(m_1+m_2)}}\sin\omega t$$

2-6

$$\omega_n^2 = \frac{k}{m} = 4.0 \qquad \omega_n = 2.0$$

$$x = x_0\cos\omega t + \frac{v_0}{\omega}\sin\omega t = 2\cos 2t - \frac{8}{2}\sin 2t$$

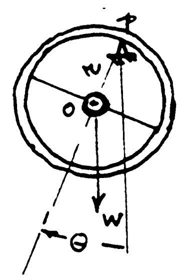
$$\dot{x} = -4\sin 2t - 8\cos 2t = 0 \qquad \therefore \tan 2t_p = -2$$

$$\therefore 2t_p = 116.57^\circ \qquad \sin 116.57^\circ = .8944 \qquad \cos 116.57 = -.4472$$

$$x_{max} = 2(-.4472) - 4(.8944) = -4.472 \text{ cm}$$

$$\ddot{x}_{max} = \omega^2 x_{max} = 4(\pm 4.472) = \pm 17.89 \text{ cm/s}^2$$

2 - 7



$$J_p \ddot{\theta} = -W r \theta \qquad \ddot{\theta} = -\omega^2 \theta$$

$$J_p = \frac{Wr}{\omega^2} = \frac{70 \times 6}{\left(\frac{2\pi}{1.22}\right)^2} = 15.83$$

$$J_o = J_p - \frac{w}{g} r^2 = 15.83 - \frac{70}{386} \times 6^2$$

$$= 9.30 \text{ lb in sec}^2$$

---

2 - 8

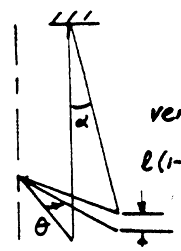
$$\omega = 2\pi \frac{53}{60} = 5.55 \text{ r/s}$$

$$J_p = \frac{Wr}{\omega^2} = \frac{21.35 \times .254}{5.55^2} = 0.1761$$

$$J_{cg} = J_o - \frac{Wr^2}{g} = 0.1761 - \frac{21.35 \times .254^2}{9.81} = 0.0356 \text{ kg m}^2$$

---

2 - 9



$$r\theta = l\alpha$$

$$\alpha = \frac{r\theta}{l}$$

vertical displ =

$$l(1-\cos\alpha) = l\left[1-\left(1-\tfrac{1}{2}\alpha^2+\cdots\right)\right]$$

$$= \frac{l\alpha^2}{2} = \frac{l}{2}\left(\frac{r\theta}{l}\right)^2$$

Work done = change in KE

$$W \frac{l}{2} \frac{r^2}{l^2} \theta_{max}^2 = \frac{1}{2} J \dot{\theta}_{max}^2 = \frac{1}{2} J \omega^2 \theta_{max}^2$$

$$J = \frac{W}{g} k^2 \qquad k = \text{rad. of gyr.}$$

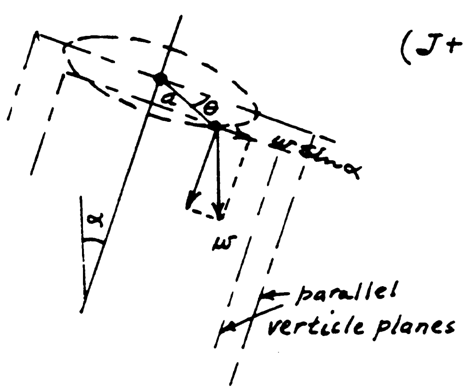
$$\frac{\cancel{W} r^2}{l} = \frac{\cancel{W}}{g} k^2 \omega^2$$

$$k^2 = \frac{r}{\omega}\sqrt{\frac{g}{l}} = \frac{.254 \times 2.17}{2\pi}\sqrt{\frac{9.81}{1.829}} = .2032$$

$$k = .4507 \text{ m}$$

---

2 - 10



Moment about shaft = $(a \sin\theta)\, w \sin\alpha$

$$\left(J + \frac{w}{g} a^2\right)\ddot{\theta} = -(a \sin\theta)\, w \sin\alpha$$

$$\cong -(a w \sin\alpha)\,\theta$$

$$\therefore f_n = \frac{1}{2\pi}\sqrt{\frac{w a \sin\alpha}{J + \frac{w}{g} a^2}}$$

2-11

$$T = \tfrac{1}{2} m \dot{x}^2 + \tfrac{1}{2} J_0 \dot{\theta}^2 \qquad r\dot{\theta} = \dot{x}$$

$$= \tfrac{1}{2}\left(m + \frac{J_0}{r^2}\right)\dot{x}^2 \qquad \dot{x} = \omega x$$

$$U = \tfrac{1}{2} k x^2 \qquad \therefore\ \omega = \sqrt{\frac{k}{m + J_0/r^2}}$$

2-12

$$\tau = 2\pi\sqrt{\frac{L}{g}} \qquad L = g\left(\frac{\tau}{2\pi}\right)^2 = 9.81\left(\frac{2}{2\pi}\right)^2 = 0.994\ m$$

$$v_{max} = L(\omega\theta_0) = \frac{.003175}{.01}\ m/s \qquad \theta_0 = \frac{.3175}{.994\,\pi} = .1017\ r = 5.826^\circ$$

2-13 water weighs $9802.\ N/m^3$ $\quad \therefore\ \rho = 1.2 \times 9802. = 11762.\ N/m^3$

buoyant force $= \pi r^2 x \cdot \rho = m\ddot{x} = \omega^2 x$

$$\frac{1}{\omega} = \frac{\tau}{2\pi} = \sqrt{\frac{m}{\pi r^2 \rho}} \qquad \begin{aligned} m &= .0372\ kg \\ r &= .0032\ m \end{aligned}$$

$$\tau = 2\pi\sqrt{\frac{.0372}{\pi \times .0032^2 \times 11762}} = 1.97\ s$$

2-14 moment about geom. center

$$-W\,8\,\theta = J_0\ddot{\theta} = -\omega^2 J_0 \theta$$

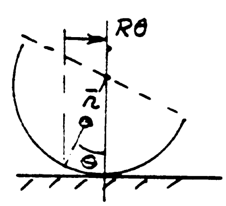
$$J_0 = \frac{8W}{\omega^2} = \frac{8W\,(1.3)^2}{(2\pi)^2} = 0.3428\ W$$

2-15

$$-W h\theta = J\ddot{\theta} = -\omega^2 J\theta$$

$$\frac{1}{\omega} = \frac{\tau}{2\pi} = \sqrt{\frac{J}{Wh}} \qquad \tau = 2\pi\sqrt{\frac{J}{Wh}}$$

2-16 Displ. of cg $= (R-\bar{r})\theta$ $\qquad T_{max} = U_{max}$



$$T_{max} = \tfrac{1}{2} m(R-\bar{r})^2\dot{\theta}_{max}^2 + \tfrac{1}{2} J_{c.g.}\dot{\theta}_{max}^2$$

$$= \tfrac{1}{2} m\left[(R-\bar{r})^2 + (R^2-\bar{r}^2)\right]\omega^2\theta_{max}^2$$

$$U_{max} = mg\bar{r}\,(1-\cos\theta_{max}) \cong mg\bar{r}\,\frac{\theta_{max}^2}{2}$$

2-16 cont $\quad T_{max} = U_{max}$

$$\omega^2 = \frac{\bar{r}g}{(R-\bar{r})^2 + (R^2-\bar{r}^2)} = \frac{\bar{r}g}{2R(R-\bar{r})}$$

$$\tau = 2\pi\sqrt{\frac{2R(R-\bar{r})}{\bar{r}g}} \qquad \text{but } \bar{r} = \frac{2R}{\pi} \qquad \therefore\ \tau = 2\pi\sqrt{\frac{R(\pi-2)}{g}}$$

2-17

$$U = mgh(1-\cos\phi) \cong mgh\tfrac{1}{2}\phi^2 \qquad h\phi = \frac{a}{2}\theta$$

$$= mg\frac{h}{2}\left(\frac{a\theta}{2h}\right)^2 = mg\frac{a^2}{8}\frac{\theta^2}{h}$$

$$T = \frac{1}{2}\left(m\frac{L^2}{12}\right)\dot{\theta}^2 = \frac{1}{2}\left(m\frac{L^2}{12}\right)\omega^2\theta^2$$

$$T_{max} = U_{max} \qquad \therefore\ \tau = 2\pi\frac{L}{a}\sqrt{\frac{h}{3g}}$$

2-18

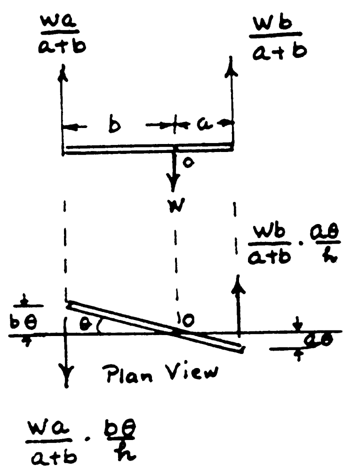
$$\tau_1 = 2\pi\sqrt{\frac{h}{g}} \qquad \text{for } \tau_2 \qquad T = \frac{1}{2}mk^2\omega^2\theta^2$$

$$U = mg\frac{L^2}{8}\frac{\theta^2}{h}$$

$$\therefore\ \tau_2 = 2\pi\sqrt{\frac{4hk^2}{gL^2}} \qquad k = \frac{\tau_2 L}{2\pi}\sqrt{\frac{g}{4h}} = \frac{\tau_2}{\tau_1}\left(\frac{L}{2}\right)$$

2-19

$$\Sigma M_o = J_{cg}\ddot{\theta} = -\frac{Wba}{a+b}\frac{\theta}{h}\cdot a - \frac{Wab}{a+b}\frac{\theta}{h}\cdot b$$



$$\frac{W}{g}k^2\ddot{\theta} + \frac{Wab}{h}\theta = 0$$

$$\ddot{\theta} + \left(\frac{gab}{k^2h}\right)\theta = 0$$

$$\therefore\ f = \frac{1}{2\pi}\sqrt{\frac{gab}{k^2h}}$$

$$\therefore\ \Sigma F_{horiz} = 0$$

2 – 20

J_o of wheel about torsion bar $= J_{cm} + m(24'')^2$

$$= m(k^2 + 24^2) = m(9^2 + 24^2) = 657\,m$$

Stiffness of torsion bar $K = \dfrac{GI_p}{l}$

$$I_p = \frac{\pi D^4}{32} = \frac{\pi}{32}(1.50)^4 = 0.497 \text{ in}^4$$ = polar mom. inertia of torsion bar

$G = 11.2 \times 10^6$ lb/in² = shear modulus of steel

$$K = \frac{(11.2 \times 10^6) \times (0.497)}{50} = 0.1113 \times 10^6 \text{ lb.in./rad.}$$

$$J_o \ddot{\theta} + K\theta = 0 \qquad f = \frac{1}{2\pi}\sqrt{\frac{K}{J_o}} = \frac{1}{2\pi}\sqrt{\frac{0.1113 \times 10^6 \times 386}{38 \times 657}} = 6.60 \text{ cps}$$ locked wheel.

With wheel free $J_o = m(24)^2 = 576\,m$

$$f = \frac{1}{2\pi}\sqrt{\frac{0.1113 \times 10^6 \times 386}{38 \times 576}} = 7.05 \text{ cps.}$$

---

2 – 21

$$\frac{l\rho}{g}\ddot{x} = -2x\rho \qquad \ddot{x} + \frac{2g}{l}x = 0$$

$$\omega^2 = \frac{2g}{l} \qquad \tau = 2\pi\sqrt{\frac{l}{2g}}$$

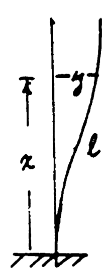
---

2 – 22

$$k = 2\left(\frac{12EI}{l^3}\right) \qquad \tau = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{ml^3}{24EI}}$$

---

2 – 23



$$y = \tfrac{1}{2}\, y_{max}\left(1 - \cos\frac{\pi x}{l}\right)\sin\omega t$$

$$\dot{y} = \tfrac{1}{2}\,\omega\, y_{max}\left(1 - \cos\frac{\pi x}{l}\right)\cos\omega t$$

$$T = \frac{1}{2}\int_0^l m(x)\,\dot{y}^2\,dx = \frac{1}{2}\,\frac{m}{4}\,y_{max}^2\int_0^l\left(1 - \cos\frac{\pi x}{l}\right)^2\omega^2\,dx\,\cos^2\omega t$$

$$= \frac{1}{2}\cdot\frac{m}{4}\,y_{max}^2\,\omega^2\cos^2\omega t\int_0^l\left(1 - 2\cos\frac{\pi x}{l} + \cos^2\frac{\pi x}{l}\right)dx$$

$$= \quad '' \quad \left[x - \frac{2l}{\pi}\sin\frac{\pi x}{l} + \frac{x}{2} + \frac{1}{2}\cdot\frac{l}{2\pi}\sin\frac{2\pi x}{l}\right]_0^l$$

2-23 Cont.

$$T = \quad '' \quad \left[\tfrac{3}{2}\ell - 0 + 0\right]$$

$$= \frac{1}{2}\left(\frac{m}{4}\cdot\frac{3\ell}{2}\right)\omega^2 y_{max}^2 \cos^2\omega t$$

$\therefore\ m_{eff} = \left(\frac{3}{8}\,m\ell\right)$ for each column, where $m\ell$ = total mass of each column

---

2-24

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J\left(\frac{\dot{x}}{b}\right)^2$$

where J = moment of inertia of Linkage about pivot.

$$x = \frac{b}{a}x_m$$

$$T = \frac{1}{2}m\left(\frac{b}{a}\right)^2\dot{x}_m^2 + \frac{1}{2}J\left(\frac{b}{a}\right)^2\frac{1}{b^2}\dot{x}_m^2$$

$$= \frac{1}{2}\left[m\left(\frac{b}{a}\right)^2 + \frac{J}{a^2}\right]\dot{x}_m^2 \qquad \therefore\ m_{eff} = \left[m\left(\frac{b}{a}\right)^2 + \frac{J}{a^2}\right]$$

2-25

$$T = \frac{1}{2}\left[J_0\dot{\theta}^2 + m_1(b\dot{\theta})^2\right] = \frac{1}{2}\left[J_0 + m_1 b^2\right]\dot{\theta}^2$$

$$\dot{\theta} = \dot{x}/b \qquad T = \frac{1}{2}\left[J_0/b^2 + m_1\right]\dot{x}^2 \qquad m_{eff} = J_0/b^2 + m_1$$

2-26 The kinetic energy is

$$T = \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}m_w(b\dot{\theta})^2 + \frac{1}{2}\left(\frac{m_s}{3}\right)(b\dot{\theta})^2 = \frac{1}{2}\left(J + m_w b^2 + \frac{1}{3}m_s b^2\right)\dot{\theta}^2$$

With the velocity at A equal to $\dot{x} = a\dot{\theta}$, T becomes

$$T = \frac{1}{2}\left(\frac{J + m_w b^2 + \frac{1}{3}m_s b^2}{a^2}\right)\dot{x}^2$$ and the effective mass at A is

$$m_A = \left(\frac{J + m_w b^2 + \frac{1}{3}m_s b^2}{a^2}\right)$$

2-27 $$\dot{y} = \frac{1}{2}\dot{y}_0\left[3\left(\frac{x}{\ell}\right)^2 - \left(\frac{x}{\ell}\right)^3\right]$$

$$T = \frac{1}{2}m\int_0^\ell \dot{y}^2\,dx = \frac{1}{2}\frac{m}{4}\dot{y}_0^2\int_0^\ell\left[9\left(\frac{x}{\ell}\right)^4 - 6\left(\frac{x}{\ell}\right)^5 + \left(\frac{x}{\ell}\right)^6\right]dx$$

$$= \frac{1}{2}m\dot{y}_0^2\frac{\ell}{4}\left[\frac{9}{5} - 1 + \frac{1}{7}\right] = \frac{1}{2}\left(\frac{33}{140}m\ell\right)\dot{y}_0^2$$

2-28 $\quad y(x) = \frac{wl}{24}\frac{l^3}{EI}\left[\left(\frac{x}{l}\right)^4 - 4\left(\frac{x}{l}\right) + 3\right]$ *x measured from free end.*

$$y_0 = y_{max} = \frac{wl}{8}\frac{l^3}{EI} \quad \therefore \quad y(x) = \frac{1}{3}\,y_{max}\left[\left(\frac{x}{l}\right)^4 - 4\left(\frac{x}{l}\right) + 3\right]$$

$$T = \frac{1}{2}\,m\int_0^l \dot{y}^2(x)\,dx = \frac{1}{2}\,m\,\frac{1}{9}\,\dot{y}_{max}^2\int_0^l\left[\left(\frac{x}{l}\right)^4 - 4\left(\frac{x}{l}\right) + 3\right]^2 dx$$

$$\int_0^l\left[\left(\frac{x}{l}\right)^4 - 4\left(\frac{x}{l}\right) + 3\right]^2 dx = \int_0^l\left[\left(\frac{x}{l}\right)^8 - 8\left(\frac{x}{l}\right)^5 + 6\left(\frac{x}{l}\right)^4 + 16\left(\frac{x}{l}\right)^2 - 24\left(\frac{x}{l}\right) + 9\right]dx$$

$$= l\left[\frac{1}{9}\left(\frac{x}{l}\right)^9 - \frac{8}{6}\left(\frac{x}{l}\right)^6 + \frac{6}{5}\left(\frac{x}{l}\right)^5 + \frac{16}{3}\left(\frac{x}{l}\right)^3 - \frac{24}{2}\left(\frac{x}{l}\right)^2 + 9\left(\frac{x}{l}\right)\right]_0^l = \frac{624}{270}\,l$$

$$\therefore\ T = \frac{1}{2}\,m\,\frac{1}{9}\,\dot{y}_{max}^2\left(\frac{624}{270}\right)l = \frac{1}{2}\left(.2568\ ml\right)\dot{y}_{max}^2$$

Compare with Prob. 2-27

$$T = \frac{1}{2}\left(\frac{33}{140}\right)ml\,\dot{y}_0^2 = \frac{1}{2}\left(.2357\ ml\right)\dot{y}_{max}^2$$

Effective mass for the two problems are nearly equal

2-29 $\quad$ Let θ_0 = rotation of J, $\quad T_0$ = total torque

T_L = torque to left of J

T_R = " " right " "

$$\left(\frac{1}{K_1} + \frac{1}{K_2}\right)T_L = \theta_0 \qquad \left(\frac{1}{K_2}\right)T_R = \theta_0$$

2-29 Cont.

$$T_o = T_L + T_R = \left[\frac{1}{\left(\frac{1}{K_1}+\frac{1}{K_2}\right)} + \frac{1}{\left(\frac{1}{K_2}\right)}\right]\theta_o = K\,\theta_o$$

$$\therefore\ K = \left(\frac{K_1 K_2}{K_1+K_2} + K_2\right) \qquad \omega_n = \sqrt{\frac{K}{J}} = \frac{2\pi}{\tau}$$

2-30

$$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} J_2\left(\frac{\dot{x}}{R_2}\right)^2 + \frac{1}{2} J_3\left(\frac{R_2}{R_1}\frac{\dot{x}}{R_2}\right)^2$$

$$= \frac{1}{2}\left[m_1 + J_2/R_2^2 + J_3/R_1^2\right]\dot{x}^2 = \frac{1}{2} m_{eff}\,\dot{x}^2$$

$$U = \frac{1}{2} k_1 x^2 + \frac{1}{2} K_3\left(\frac{x}{R_1}\right)^2 = \frac{1}{2}\left[k_1 + K_3/R_1^2\right]x^2 = \frac{1}{2} k_{eff}\,x^2$$

2-31

$$T = \frac{1}{2} J_1 \dot{\theta}_1^2 + \frac{1}{2} J_2\left(\frac{r_1}{r_2}\dot{\theta}_1\right)^2$$

$$= \frac{1}{2}\left[J_1 + J_2\left(\frac{r_1}{r_2}\right)^2\right]\dot{\theta}_1^2 = \frac{1}{2} J_{eff}\,\dot{\theta}_1^2$$

2-32

$$T = \frac{1}{2} J_1 \dot{\theta}_1^2 + \frac{1}{2}(m_o + m_2)\dot{x}^2 + \frac{1}{2} J_2 \dot{\theta}_2^2$$

$$\dot{\theta}_1 = \dot{x}/r_1 \qquad \dot{\theta}_2 = \dot{x}/r_2$$

$$T = \frac{1}{2}\left[J_1/r_1^2 + (m_o + m_2) + J_2/r_2^2\right]\dot{x}^2$$

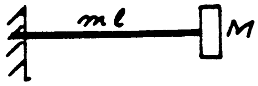
$$U = \frac{1}{2} k_1 (R\theta_1)^2 + \frac{1}{2} k_2 (r_2\theta_2)^2$$

$$= \frac{1}{2} k_1\left(\frac{R}{r_1}x\right)^2 + \frac{1}{2} k_2\left(r_2\frac{x}{r_2}\right)^2$$

$$= \frac{1}{2}\left[k_1\left(\frac{R}{r_1}\right)^2 + k_2\right]x^2$$

$$f = \frac{1}{2\pi}\sqrt{\frac{k_1\left(\frac{R}{r_1}\right)^2 + k_2}{J_1/r_1^2 + (m_o+m_1) + J_2/r_2^2}}$$

2 − 33 $\qquad m_{eff} = M + \frac{33}{140} m\ell \qquad$ (see Prob. 2-27)



beam vol $= (.1016 \times .635 \times 8.89) = .5735 \ \text{cm}^3$

wt. of steel $= 0.07655 \ \text{N/cm}^3$

wt. of beam $= .5735 \times .07655 = .04390 \ \text{N}$

mass of beam $= \frac{.04390}{9.81} = .00475 \ \text{kg} = m\ell$

$\frac{33}{140} m\ell = .001055 \qquad$ beam stiffness $= \frac{3EI}{\ell^3} = k$

$E = 200 \times 10^9 \ \text{N/m}^2 \qquad I = \frac{bh^3}{12} = \frac{.635 \times .1016^3}{12} = .0000553 \times 10^{-8} \ \text{m}^4$

$$k = \frac{3 \times 200 \times 10^9 \times 555 \times 10^{-15}}{(.0889)^3} = 473.96 \ \text{N/m}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m_{eff}}} \qquad m_{eff} = M + .001055 = \frac{3EI}{\ell^3 4\pi^2 f^2} = \frac{473.96}{4\pi^2 \times 400}$$

$\therefore M = 0.0289 \ \text{kg}$

---

2 − 34 $\qquad c_c = 2\sqrt{mk} = 2\sqrt{.907 \times 7. \times 10^2} = 50.4 \ \frac{\text{Ns}}{\text{m}}$

---

2 − 35 $\qquad F_d = cv \qquad c = \frac{F_d}{v} = \frac{0.50}{1.20} = 0.417 \ \frac{\text{lb sec}}{\text{in}}$

$$c = 0.417 \ \frac{\text{lb.s}}{\text{in}} \times 4.448 \ \frac{\text{N}}{\text{lb}} \times \frac{1}{2.54} \ \frac{\text{in}}{\text{cm}} = 0.7303 \ \frac{\text{NS}}{\text{cm}} = 73.03 \ \frac{\text{NS}}{\text{m}}$$

$$\zeta = \frac{c}{c_c} = \frac{73.03}{50.4} = 1.45$$

---

2 − 36

(a) $\zeta = 2. \qquad$ Eq 2.6-20, 2.6-21 $\qquad A = -B = \frac{v_0}{2\omega_n\sqrt{\zeta^2 - 1}}$

$$\frac{x\omega_n}{v_0} = \frac{1}{3.464}\left(e^{-0.268\omega_n t} - e^{-3.732\,\omega_n t}\right)$$

(b) $\zeta = 0.50 \qquad \frac{x\omega_n}{v_0} = \frac{e^{-0.50\omega_n t}}{0.865} \sin 0.865\omega_n t \qquad$ Eq 2.6-17

(c) $\zeta = 1.0 \qquad \frac{x\omega_n}{v_0} = \omega_n t \ e^{-\omega_n t}$

Eq 2.6-23

2-37 Solve on computer using RUNGA. With ω_n not given, the equation

$$\frac{d^2x}{dt^2} + 2\zeta \frac{dx}{dt} + \omega_n^2 x = 0$$

can be rewritten for RUNGA as follows

Let $T = \omega_n t$ and $y = \frac{x\omega_n}{v_0}$

Then $x = \frac{v_0}{\omega_n} y$, $\frac{dx}{dt} = \frac{v_0}{\omega_n}\frac{dy}{dt} = v_0 \frac{dy}{dT}$

$$\frac{d^2x}{dt^2} = \frac{v_0}{\omega_n}\frac{d^2y}{dt^2} = \frac{v_0}{\omega_n}\frac{\omega_n^2}{\omega_n^2}\frac{d^2y}{dt^2} = v_0\omega_n \frac{d^2y}{dT^2}$$

The original equation then becomes

$$v_0\omega_n\left[\frac{d^2y}{dT^2} + 2\zeta\frac{dy}{dT} + y\right] = 0$$

and $v_0 = 1.0$ for all computations.

Computer solutions for (a), (b) and (c) are:

(a) $\zeta = 2.0$ $\quad y_{max} = \frac{\omega_n x_{max}}{v_0} = 0.1953$ (critical damping)

(b) $\zeta = .50$ $\quad y_{max} = 0.534$ (oscillatory)

(c) $\zeta = 1.0$ $\quad y_{max} = 0.3666$ (aperiodic)

See Prob 2-36 for analytic solution:

```
» 237 Cont                                                    Prob. 237(a)
Enter the value of the mass (kg) : 1

m =

     1


Enter the value of the damping coefficient (N.s/m) : 4

c =

     4


Enter the value of the spring constant (N/m) : 1

k =

     1


Enter the value of the initial position (m) : 0

x1 =

     0


Enter the value of the initial velocity (m/s) : 1

y1 =

     1


ans =

Natural period

T =

    6.2832
```

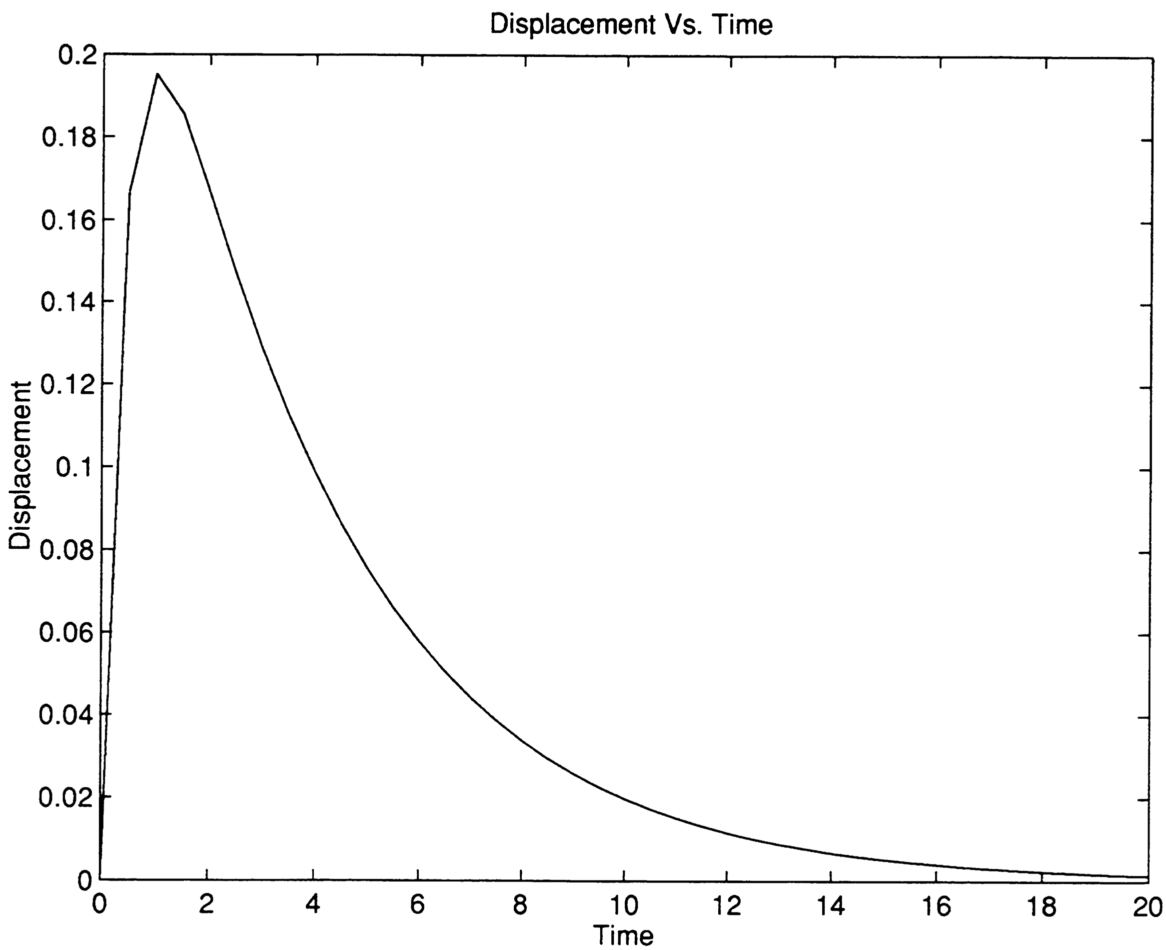
2-37 cont

```
 Enter the value of time increment in seconds ( < T as above ) : .5

dt =

    0.5000


 Enter the value of the initial time in seconds (0) : 0

t1 =

     0


 Enter the value of the final time in seconds : 10

tf =

    10


ans =

     Time          Displ.       Vel.


ans =

         0         0    1.0000
    0.5000    0.1667    0.2526
    1.0000    0.1953    0.0360
    1.5000    0.1856   -0.0235
    2.0000    0.1667   -0.0368
    2.5000    0.1471   -0.0371
    3.0000    0.1290   -0.0339
    3.5000    0.1130   -0.0301
    4.0000    0.0988   -0.0264
    4.5000    0.0864   -0.0231
    5.0000    0.0756   -0.0203
    5.5000    0.0661   -0.0177
    6.0000    0.0578   -0.0155
    6.5000    0.0506   -0.0136
    7.0000    0.0442   -0.0119
```

2-37 cont

7.5000	0.0387	-0.0104
8.0000	0.0338	-0.0091
8.5000	0.0296	-0.0079
9.0000	0.0259	-0.0069
9.5000	0.0226	-0.0061
10.0000	0.0198	-0.0053

```
ans =

maximum amplitude


A =

    0.1953
```

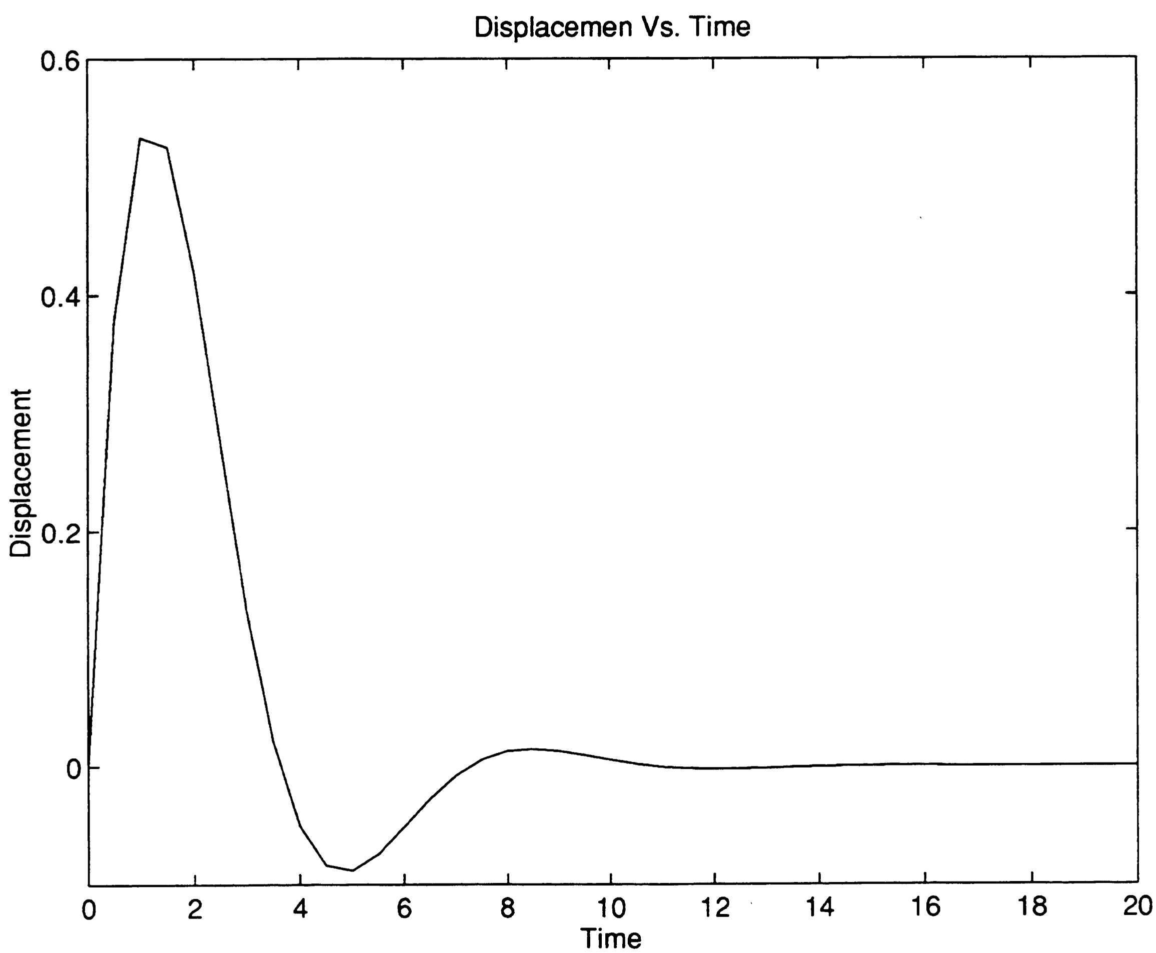


» 2 - 37 Cont 2.37(b)

```
Enter the value of the mass (kg) : 1

m =

     1


Enter the value of the damping coefficient (N.s/m) : 1

c =

     1


Enter the value of the spring constant (N/m) : 1

k =

     1


Enter the value of the initial position (m) : 0

x1 =

     0
Enter the value of the initial velocity (m/s) : 1

y1 =

     1


ans =

Natural period


T =

    6.2832
```

2-37 Cont

```
Enter the value of time increment in seconds ( < T as above ) : .5

dt =

    0.5000


Enter the value of the initial time in seconds (0) : 0

t1 =

     0


Enter the value of the final time in seconds : 20

tf =

    20
```



2-5t Cont

Time	Displ.	Vel.
0	0	1.0000
0.5000	0.3776	0.5182
1.0000	0.5340	0.1260
1.5000	0.5259	-0.1363
2.0000	0.4196	-0.2692
2.5000	0.2743	-0.2980
3.0000	0.1332	-0.2580
3.5000	0.0219	-0.1840
4.0000	-0.0499	-0.1036
4.5000	-0.0838	-0.0349
5.0000	-0.0882	0.0136
5.5000	-0.0739	0.0404
6.0000	-0.0510	0.0488
6.5000	-0.0272	0.0446
7.0000	-0.0076	0.0334
7.5000	0.0058	0.0202
8.0000	0.0128	0.0082
8.5000	0.0146	-0.0006
9.0000	0.0129	-0.0058
9.5000	0.0093	-0.0079
10.0000	0.0054	-0.0076
10.5000	0.0020	-0.0060
11.0000	-0.0005	-0.0038
11.5000	-0.0019	-0.0018
12.0000	-0.0024	-0.0002
12.5000	-0.0022	0.0008
13.0000	-0.0017	0.0012
13.5000	-0.0010	0.0013
14.0000	-0.0004	0.0011
14.5000	-0.0000	0.0007
15.0000	0.0003	0.0004
15.5000	0.0004	0.0001
16.0000	0.0004	-0.0001
16.5000	0.0003	-0.0002
17.0000	0.0002	-0.0002
17.5000	0.0001	-0.0002
18.0000	0.0000	-0.0001
18.5000	-0.0000	-0.0001
19.0000	-0.0001	-0.0000
19.5000	-0.0001	0.0000
20.0000	-0.0001	0.0000

2 - 37 Cont

```
maximum amplitude

A =

    0.5340
```

» prob 2.37 (a)

```
 Enter the value of the mass (kg) : 1

m =

     1


 Enter the value of the damping coefficient (N.s/m) : 2

c =

     2


 Enter the value of the spring constant (N/m) : 1

k =

     1


 Enter the value of the initial position (m) : 0

x1 =

     0


 Enter the value of the initial velocity (m/s) : 1

y1 =

     1


ans =
```

2-37 cont

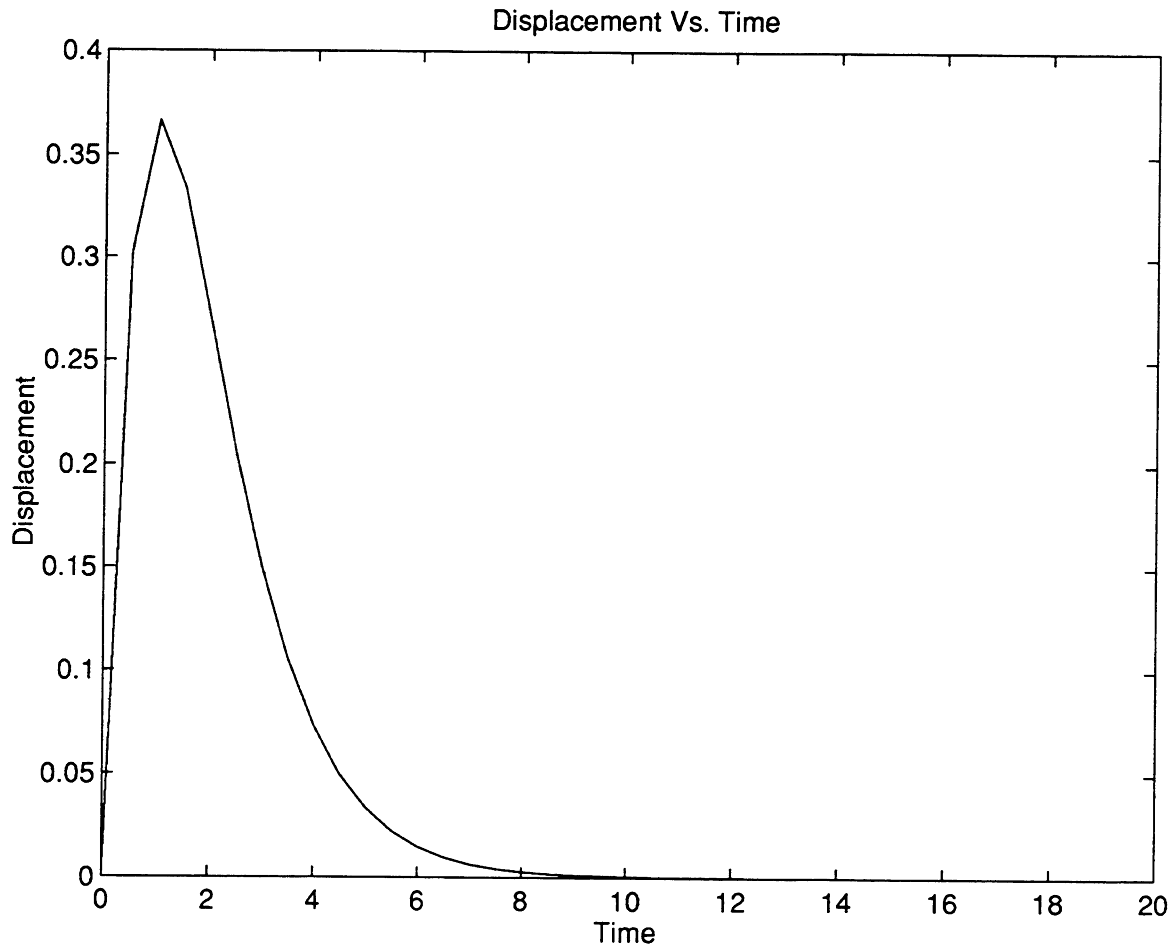
```
Natural period

T =

    6.2832


 Enter the value of time increment in seconds ( < T as above ) : .5

dt =

    0.5000


 Enter the value of the initial time in seconds (0) : 0

t1 =

     0


 Enter the value of the final time in seconds : 20

tf =

    20


ans =

     Time          Displ.        Vel.


ans =

         0         0    1.0000
    0.5000    0.3021    0.3047
    1.0000    0.3666    0.0016
    1.5000    0.3337   -0.1103
    2.0000    0.2699   -0.1344
    2.5000    0.2047   -0.1225
    3.0000    0.1491   -0.0992
    3.5000    0.1055   -0.0752
```

2-37 cont

4.0000	0.0732	-0.0548
4.5000	0.0500	-0.0388
5.0000	0.0337	-0.0269
5.5000	0.0225	-0.0184
6.0000	0.0149	-0.0124
6.5000	0.0098	-0.0083
7.0000	0.0064	-0.0055
7.5000	0.0042	-0.0036
8.0000	0.0027	-0.0024
8.5000	0.0017	-0.0015
9.0000	0.0011	-0.0010
9.5000	0.0007	-0.0006
10.0000	0.0005	-0.0004
10.5000	0.0003	-0.0003
11.0000	0.0002	-0.0002
11.5000	0.0001	-0.0001
12.0000	0.0001	-0.0001
12.5000	0.0000	-0.0000
13.0000	0.0000	-0.0000
13.5000	0.0000	-0.0000
14.0000	0.0000	-0.0000
14.5000	0.0000	-0.0000
15.0000	0.0000	-0.0000
15.5000	0.0000	-0.0000
16.0000	0.0000	-0.0000
16.5000	0.0000	-0.0000
17.0000	0.0000	-0.0000
17.5000	0.0000	-0.0000
18.0000	0.0000	-0.0000
18.5000	0.0000	-0.0000
19.0000	0.0000	-0.0000
19.5000	0.0000	-0.0000
20.0000	0.0000	-0.0000

```
ans =

maximum amplitude


A =

    0.3666

»
```

2-5+ cont



Displacement Vs. Time
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
Displacement
0
2
4
6
8
10
12
14
16
18
20
Time

2-38

$$\delta = \ln\frac{x_1}{x_2} = \ln\frac{1.00}{.980} = \ln 1.020408 = 0.0202$$

$$\zeta \cong \frac{\delta}{2\pi} = .003215$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1750}{2.267}} = 27.78 \cong \omega_d$$

$$c = 2m\omega_n\zeta = 2 \times 2.267 \times 27.78 \times .003215 = .405 \ \frac{Ns}{m}$$

2-39

(a) $$\zeta = \frac{c}{2m}\sqrt{\frac{m}{k}} = \frac{12.43}{2 \times 4.534}\sqrt{\frac{4.534}{3500}} = 0.0493$$

(b) $$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = 0.3101$$

(c) $$\frac{x_n}{x_{n+1}} = e^{\delta} = (2.718)^{.3101} = 1.364$$

2-40

(a) $$\zeta = \frac{c}{2\sqrt{mk}} = \frac{70}{2\sqrt{17.5 \times 7000}} = 0.10$$

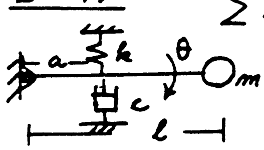
(b) $$f_d = \frac{1}{2\pi}\sqrt{1-\zeta^2}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{1-.01}\sqrt{\frac{7000}{17.5}} = 3.167 \text{ Hz}$$

(c) $$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = .6315$$

(d) $$x_n / x_{n+1} = e^{.6315} = 1.874$$

2-41

$$\sum M_0 = -a\,c(a\dot\theta) - a\,k(a\theta) = m\ell^2\ddot\theta$$



$$\ddot\theta + \frac{c}{m}\left(\frac{a}{\ell}\right)^2\dot\theta + \frac{k}{m}\left(\frac{a}{\ell}\right)^2\theta = 0 \qquad \text{let } \theta = e^{st}$$

$$s_{1,2} = -\frac{c}{2m}\left(\frac{a}{\ell}\right)^2 \pm \sqrt{\left(\frac{ca^2}{2m\ell^2}\right)^2 - \frac{k}{m}\left(\frac{a}{\ell}\right)^2}$$

crit. damp. $$\frac{c_c a^2}{2m\ell^2} = \frac{a}{\ell}\sqrt{\frac{k}{m}} \qquad c_c = 2\frac{\ell}{a}\sqrt{km}$$

$$\omega_d = \frac{a}{\ell}\sqrt{\frac{k}{m} - \left(\frac{ca}{2m\ell}\right)^2} = \frac{a}{\ell}\sqrt{\frac{k}{m}}\sqrt{1 - \left(\frac{ca}{2\ell\sqrt{km}}\right)^2} = \omega_n\sqrt{1-\zeta^2}$$

$$\therefore\ \omega_n = \frac{a}{\ell}\sqrt{\frac{k}{m}}, \quad \zeta = \frac{ca}{2\ell\sqrt{km}}$$ identify from $\ddot\theta + 2\zeta\omega_n\dot\theta + \omega_n^2\theta = 0$

2-42

$$\Sigma M_0 = m a^2 \ddot{\theta} = -k b^2 \theta - c a^2 \dot{\theta}$$

$$\ddot{\theta} + \frac{c}{m}\dot{\theta} + \frac{k}{m}\left(\frac{b}{a}\right)^2 \theta = 0$$

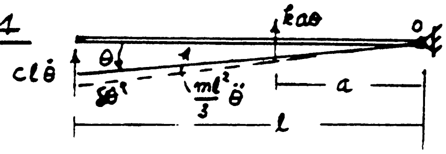
$$\therefore \quad \omega_n = \frac{b}{a}\sqrt{\frac{k}{m}} \qquad \omega_d = \sqrt{\frac{k}{m}\left(\frac{b}{a}\right)^2 - \left(\frac{c}{2m}\right)^2} \qquad c_c = \frac{2b}{a}\sqrt{km}$$

2-43

$$\delta = \ln\frac{1.0}{0.95} = \ln 1.0527 = .05129$$

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = .05129 \qquad \zeta = .00816$$

2-44



$$\delta W = -\frac{ml^2}{3}\ddot{\theta}\,\delta\theta - cl\dot{\theta}\,l\,\delta\theta - ka\theta\,a\,\delta\theta = 0$$

$$\ddot{\theta} + \frac{3c}{m}\dot{\theta} + \frac{3k}{m}\left(\frac{a}{l}\right)^2\theta = 0$$

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = 0$$

$$\therefore \omega_n = \frac{a}{l}\sqrt{\frac{3k}{m}}, \qquad c_c = \frac{2a}{3l}\sqrt{3km}, \qquad \zeta = \frac{3}{2}\frac{c}{m\omega_n}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \frac{a}{l}\sqrt{\frac{3k}{m}}\sqrt{1-\frac{9}{4}\left(\frac{c}{m\omega_n}\right)^2} = \frac{a}{l}\sqrt{\frac{3k}{m}}\sqrt{1-\frac{3}{4km}\left(\frac{cl}{a}\right)^2}$$

2-45

$$\frac{W}{g}\ddot{x} + 2\mu A\dot{x} + kx = 0$$

$$\ddot{x} + \frac{2\mu A g}{W}\dot{x} + \frac{kg}{W}x = 0 \qquad \therefore \tau_1 = 2\pi\sqrt{\frac{W}{kg}}$$

$$f_2 = \frac{1}{\tau_2} = \frac{1}{2\pi}\sqrt{\frac{kg}{W} - \left(\frac{\mu A g}{W}\right)^2} = \frac{1}{2\pi}\sqrt{\left(\frac{2\pi}{\tau_1}\right)^2 - \left(\frac{\mu A g}{W}\right)^2}$$

square both sides

$$\left(\frac{2\pi}{\tau_2}\right)^2 - \left(\frac{2\pi}{\tau_1}\right)^2 = -\left(\frac{\mu A g}{W}\right)^2$$

$$\therefore \quad \mu = \frac{2\pi W}{Ag}\sqrt{\frac{\tau_2^2 - \tau_1^2}{\tau_1^2\,\tau_2^2}} = \frac{2\pi W}{Ag\,\tau_1\tau_2}\sqrt{\tau_2^2 - \tau_1^2}$$

2-46

$$\omega_n = \sqrt{\frac{20,000 \times 32.2}{1200}} = 23.17 \text{ r/s}$$

$$\frac{1}{2} m \dot{x}_{max}^2 = \frac{1}{2} k x_{max}^2 \qquad \dot{x}_{max} = 23.17 \times 4 = 92.66 \text{ ft/s}.$$

Eq. (2.3-19) $\quad x = e^{-\omega_n t}[0 + \omega_n x(0)]t + x(0) e^{-\omega_n t}$

$$= e^{-\omega_n t} x(0) [1 + \omega_n t]$$

$$\frac{2}{12}' = e^{-\omega_n t} 4[1 - \omega_n t] \quad \text{or} \quad e^{-\omega_n t}[1 + \omega_n t] = .0417$$

solve by trial

$\omega_n t$	$e^{-\omega_n t}$	$e^{-\omega_n t}[1 + \omega_n t]$	
4.90	.00745	.0439	
4.96	.007017	.04182	← close
4.97	.006947	.04147	

$$\therefore \omega_n t = 4.96 \qquad t = \frac{4.96}{23.17} = 0.214 \text{ s}$$

2-47

$$\omega_n = \sqrt{\frac{35000}{4.53}} = 87.89 \text{ r/s}$$

$$\tau = \frac{2\pi}{87.89} = .0715 \text{ s} \qquad c_c = 2\sqrt{km} = 797.04$$

$$\zeta = .2197$$

$$\tau_d = \sqrt{1-\zeta^2}\,\tau = .0697$$

Eq. (2.6-16)

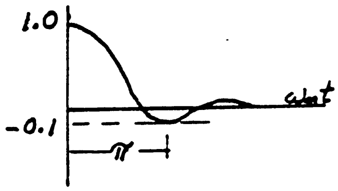
$$x = \frac{\dot{x}(0)}{\omega_n \sqrt{1-\zeta^2}} e^{-\zeta \omega_n t} \sin\sqrt{1-\zeta^2}\,\omega_n t$$

at x_{max}, $\sin\sqrt{1-\zeta^2}\,\omega_n t \cong 1.0$, also $\omega_n t = \pi/2$

$$x = \frac{15.24}{87.89 \times .9756} e^{-.2197(\pi/2)} = .1259 \text{ m}$$

$$t = \frac{1}{4}\tau_d = .0174 \text{ s}.$$

2-48



Eq. (2.6-17) for $\dot{x}(0) = 0$

$$x = x(0)\, e^{-\zeta\omega_n t} \cos\sqrt{1-\zeta^2}\,\omega_n t$$

at $\sqrt{1-\zeta^2}\,\omega_n t = \pi \qquad \cos\sqrt{1-\zeta^2}\,\omega_n t = -1$

$$-.10 = 1\, e^{-\zeta\omega_n t}(-1) \qquad \text{solve by trial}$$

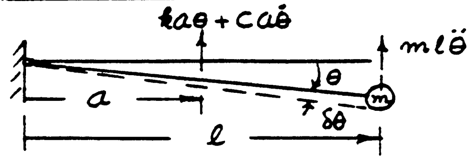
ζ	$\sqrt{1-\zeta^2}$	$\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}$	$e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$
.50	.866		.1630
.59	.8074	−2.2957	.1007 ←
.60	.800	−2.3562	.0948

$$\therefore\ \zeta_1 = 0.59$$

If $\zeta = \frac{1}{2}\zeta_1 = 0.295, \quad \sqrt{1-\zeta^2} = .9555$

$$x_{overshoot} = 1\, e^{\frac{-.295\pi}{.9555}} = 0.379 = 37.9\%$$

2-49(a) (Prob. 2-41 by V.W.)



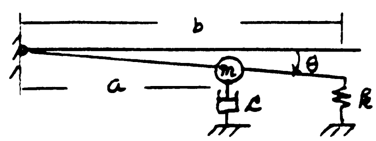
$$\delta W = -m\ell\ddot{\theta}\cdot\ell\delta\theta - (ka\theta + ca\dot{\theta})\,a\delta\theta = 0$$

$$\ddot{\theta} + \frac{ca^2}{m\ell^2}\dot{\theta} + \frac{ka^2}{m\ell^2}\theta = 0$$

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = 0 \qquad \therefore\ \omega_n = \frac{a}{\ell}\sqrt{\frac{k}{m}}, \quad \zeta = \frac{1}{2}\frac{c}{m}\left(\frac{a}{\ell}\right)^2\cdot\frac{\ell}{a}\sqrt{\frac{m}{k}} = \frac{ca}{2\ell\sqrt{km}}$$

$$\omega_d = \frac{a}{\ell}\sqrt{\frac{k}{m}}\sqrt{1 - \frac{1}{km}\left(\frac{ca}{2\ell}\right)^2}$$

2-49(b) (Prob. 2-42 by V.W.)



$$\delta W = -ma\ddot{\theta}\cdot a\delta\theta - ca\dot{\theta}\cdot a\delta\theta - kb\theta\cdot b\delta\theta = 0$$

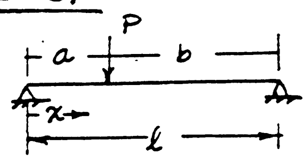
$$\therefore\ \ddot{\theta} + \frac{c}{m}\dot{\theta} + \frac{k}{m}\left(\frac{b}{a}\right)^2\theta = 0$$

2-50

$k_{eff} = (k_1 + k_2)$ in series with k_3

$$= \frac{(k_1 + k_2)\,k_3}{k_1 + k_2 + k_3}$$

2-51



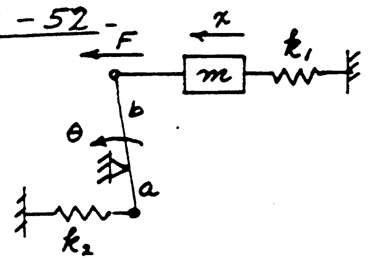
$$y(x) = \frac{Pbx}{6EIl}\left(l^2 - x^2 - b^2\right) \qquad 0 \le x \le a$$

Let $x = \frac{l}{3}$.

$$y\left(\tfrac{l}{3}\right) = \frac{P\frac{2}{3}l \cdot \frac{1}{3}l}{6EIl}\left(l^2 - \frac{l^2}{9} - \frac{4}{9}l^2\right) = \frac{Pl^3}{EI}\cdot\frac{4}{243}$$

$$\text{Flexibility} = \frac{y}{P} = \frac{4}{243}\,\frac{l^3}{EI} \qquad \text{at } \frac{x}{l} = \frac{1}{3}$$

2-52



$$F = k_1 b\theta + \frac{a}{b}k_2 a\theta$$

$$x = b\theta$$

$$\therefore F = k_1 x + \left(\frac{a}{b}\right)^2 k_2 x$$

$$k_{eff} = \frac{F}{x} = k_1 + \left(\frac{a}{b}\right)^2 k_2$$

2-53 $\qquad k_{eff} = \dfrac{k_1 k_2}{k_1 + k_2}$

2-54 Eq. (2.6-17) $\qquad x(t) = e^{-\zeta\omega_n t}\left(\dfrac{\zeta}{\sqrt{1-\zeta^2}}\sin\sqrt{1-\zeta^2}\,\omega_n t + \cos\sqrt{1-\zeta^2}\,\omega_n t\right)$

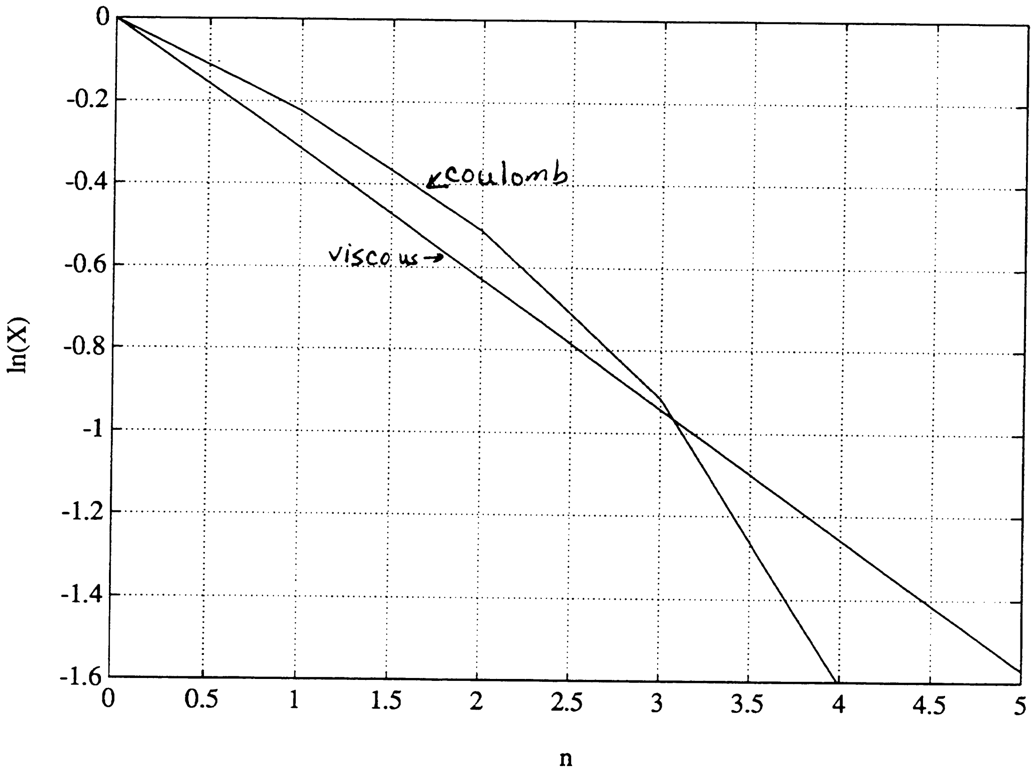
at $\omega_n t = 2\pi, 4\pi, 6\pi$, etc. $\qquad x(t) \cong e^{-\zeta\omega_n t}(0 + 1)$

n	$\omega_n t$	$e^{-.05\omega_n t}$	X_n
0	0	1.0	1.0
1	2π	.7304	.8
2	4π	.5335	.6
3	6π	.3896	.4
4	8π	.2845	.2
5	10π	.2078	0

For Coulomb friction

$$X_1 - X_2 = \frac{4F_d}{k} = \frac{4 \times .05k}{k} = .20$$

$$X_n = 1 - .2n$$



The two amplitudes are equal for $n \cong 3.1$

2 − 55

$$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} I_o \left(\frac{\dot{x}}{r}\right)^2$$

$$U = \frac{1}{2} k_1 x^2 + \frac{1}{2} k_2 \left(a\frac{x}{r} + x\right)^2$$

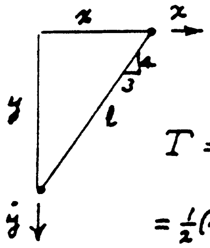
$$\frac{d}{dt}(T+U) = \left(m_1 \ddot{x} + m_2 \ddot{x} + \frac{I_o}{r^2}\ddot{x}\right)\dot{x}$$

$$+\left[k_1 + k_2\left(1+\frac{a}{r}\right)^2\right] x\,\dot{x} = -c\,\dot{x}\,\dot{x} = \frac{dW}{dt}$$

$$\left(m_1 + m_2 + I_o/r^2\right)\ddot{x} + \left[k_1 + k_2\left(1+\frac{a}{r}\right)^2\right] x + c\dot{x} = 0$$

$$c_c = 2\sqrt{k_{eff}\, m_{eff}} = 2\sqrt{\left[k_1 + k_2\left(1+\frac{a}{r}\right)^2\right]\left(m_1 + m_2 + I_o/r^2\right)}$$

2 - 56



$$x + y = \ell$$

$$2x\,dx + 2y\,dy = 0 \qquad \therefore \frac{dy}{dx} = -\frac{x}{y}$$

$$\dot{y} = -\frac{3}{4}\dot{x}$$

$$T = \frac{1}{2}(m\ell)\frac{\ell^2}{3}\left(\frac{\dot{x}}{\ell}\right)^2 + \frac{1}{2}(m\ell)\left[\left(\frac{\dot{x}}{2}\right)^2 + \left(\frac{\dot{y}}{2}\right)^2\right] + \frac{1}{2}(m\ell)\frac{\ell^2}{12}\left[\frac{4\dot{x}}{5\ell} + \frac{3}{5}\frac{\dot{y}}{\ell}\right] + \frac{1}{2}M\dot{y}^2$$

$$= \frac{1}{2}(m\ell)\left\{\frac{\ell^2}{3}\left(\frac{\dot{x}}{\ell}\right)^2 + \left[\left(\frac{\dot{x}}{2}\right)^2 + \left(\frac{3\dot{x}}{8}\right)^2\right] + \frac{\ell^2}{12}\left[\frac{4}{5}\frac{\dot{x}}{\ell} + \frac{9}{20}\frac{\dot{x}}{\ell}\right]^2 + \frac{1}{2}M\left(\frac{3\dot{x}}{4}\right)^2\right\}$$

$$= \frac{1}{2}(m\ell)\left[\frac{1}{3} + \frac{1}{4} + \frac{9}{64} + \frac{1}{12}\left(\frac{16}{25} + \frac{81}{400} + \frac{72}{100}\right)\right]\dot{x}^2 + \frac{1}{2}M\frac{9}{16}\dot{x}^2$$

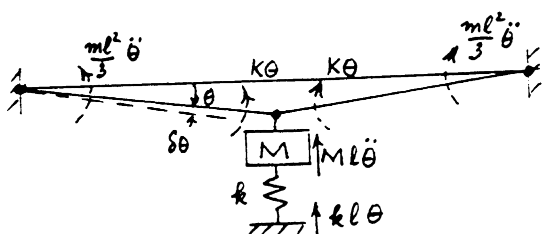
$$= \frac{1}{2}\left[(.854\,m\ell) + .5625\,M\right]\dot{x}^2$$

$$U = \frac{1}{2}k y^2 = \frac{1}{2}k\frac{9}{16}x^2$$

$$\frac{d}{dt}(T + U) = -c\,\frac{2}{3}\ell\,\frac{\dot{x}}{\ell} = -\frac{2}{3}c\dot{x}$$

$$(0.8541\,m\ell + .5625\,M)\ddot{x} + .5625\,k x + \frac{2}{3}c\dot{x} = 0$$

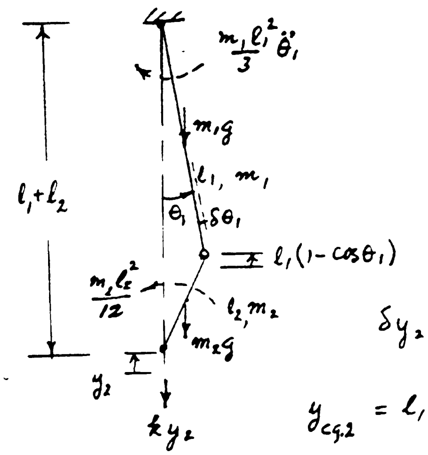
2 - 57



$$\delta W = \left[-M\ell\ddot{\theta}\,\ell\delta\theta - k\ell\theta\,\ell\delta\theta\right] - 2K\theta\cdot\delta\theta - 2\frac{m\ell^2}{3}\ddot{\theta}\,\delta\theta = 0$$

$$\left(M\ell^2 + \frac{2}{3}m\ell^2\right)\ddot{\theta} + \left(k\ell^2 + 2K\right)\theta = 0$$

2 - 58



Accel of c.g. in vertical direction $\cong 0$
but work done by gravity is not zero

$$\delta y_{cg.1} = \frac{l_1}{2}\sin\theta_1\delta\theta_1 \cong \frac{l_1}{2}\theta_1\delta\theta_1$$

$$y_2 = l_1(1-\cos\theta_1) + l_2(1-\cos\theta_2)$$

$$l_1\theta_1 \cong l_2\theta_2\,,\quad \theta_2 = \frac{l_1}{l_2}\theta_1\quad \therefore\ \delta\theta_2 = \frac{l_1}{l_2}\delta\theta_1$$

$$\delta y_2 = l_1(\theta_1 + \frac{l_1}{l_2}\theta_1)\delta\theta_1 = l(1+\frac{l_1}{l_2})\theta_1\delta\theta_1$$

$$y_{cg.2} = l_1(1-\cos\theta_1) + \frac{l_2}{2}(1-\cos\theta_2)\quad \therefore\ \delta y_{cg2} = l_1\theta_1\delta\theta_1 + \frac{l_2}{2}\theta_2\delta\theta_2$$

$$= l_1\theta_1\delta\theta_1 + \frac{l_2}{2}\frac{l_1}{l_2}\theta_1\frac{l_1}{l_2}\delta\theta_1$$

$$= l_1(1+\frac{l_1}{2l_2})\theta_1\delta\theta_1$$

$$\delta W = -(\frac{m_1 l_1^2}{3})\ddot{\theta}_1\delta\theta_1 - (\frac{m_2 l_2^2}{12}\ddot{\theta}_2)\delta\theta_2 - m_1 g\frac{l_1}{2}\theta_1\delta\theta_1 - m_2 g\, l_1(1+\frac{l_1}{2l_2})\theta_1\delta\theta_1$$
$$-k\left[l_1(1-\cos\theta_1) + l_2(1-\cos\theta_2)\right]l(1+\frac{l_1}{l_2})\theta_1\delta\theta_1 = 0$$

sub. $\ddot{\theta}_2 = \frac{l_1}{l_2}\ddot{\theta}_1\,,\quad \delta\theta_2 = \frac{l_1}{l_2}\delta\theta_1\,,\quad \theta_2 = \frac{l_1}{l_2}\theta_1$

$$\left[m_1\frac{l_1^2}{3} + \frac{m_2 l_2^2}{12}\left(\frac{l_1}{l_2}\right)^2\right]\ddot{\theta}_1 + \left[m_1 g\frac{l_1}{2} + m_2 g\, l_1(1+\frac{l_1}{2l_2})\right]\theta_1 = 0$$

(springforce is 2nd order infinitesmal)

2-59 With the initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$, the solution must be considered for each half cycle. During the first half cycle, with m moving from right to left, $\dot{x}(t)$ is negative so that $sgn(\dot{x})$ is positive and the equations of motion are;

$$x(t) = A\sin\omega_n t + B\cos\omega_n t + \frac{\mu F}{k}$$
$$\dot{x}(t) = \omega_n A\cos\omega_n t - \omega_n B\sin\omega_n t$$

At $t=0$

$$x(0) = x_0 = B + \frac{\mu F}{k} \qquad \therefore B = (x_0 - \frac{\mu F}{k})$$
$$\dot{x}(0) = 0 = \omega_n A \qquad \therefore A = 0$$

and the general solution becomes

$$x(t) = (x_0 - \frac{\mu F}{k})\cos\omega_n t + \frac{\mu F}{k}$$
$$\dot{x}(t) = -\omega_n(x_0 - \frac{\mu F}{k})\sin\omega_n t$$

For the second half cycle $\dot{x}$ is positive and the friction force is negative. The new equation must now be written with two other constants C and D

$$x(t) = C\sin\omega_n t + D\cos\omega_n t - \frac{\mu F}{k}$$
$$\dot{x}(t) = \omega_n C\cos\omega_n t - \omega_n D\sin\omega_n t$$

We again measure time $t=0$ at the beginning of the 2^{nd} half cycle, with new initial conditions found from the previous equations as $x(0) = (-x_0 + 2\frac{\mu F}{k})$ and $\dot{x}(0) = 0$

$$\therefore D = -x_0 + 3\frac{\mu F}{k} \quad \text{and} \quad C = 0$$

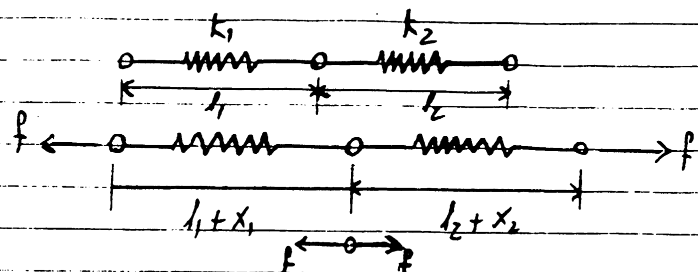
$\therefore$ The new equations for the second half cycle

$$x(t) = (-x_0 + 3\frac{\mu F}{k})\cos\omega_n t - \frac{\mu F}{k}$$
$$\dot{x}(t) = -\omega_n(-x_0 + 3\frac{\mu F}{k})\sin\omega_n t$$

2-60

unstretched

stretched



In the stretch position the total displacement of the two springs is x_1+x_2. Since each spring exhibits the same force f, we have

$$x_{eq} = x_1 + x_2$$

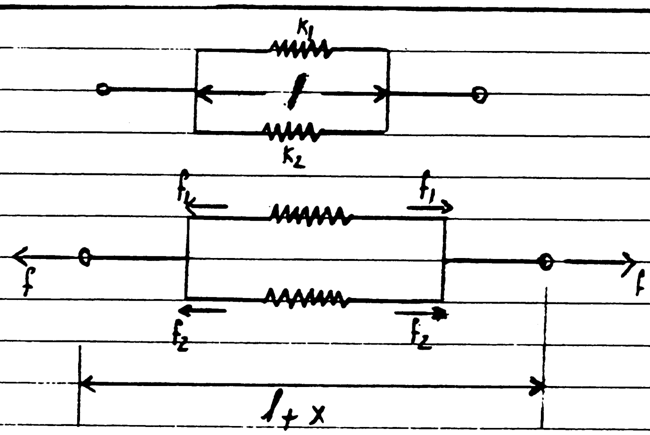
$$\therefore \frac{f}{k_{eq}} = \frac{f}{k_1} + \frac{f}{k_2} \Longrightarrow k_{eq} = \frac{1}{1/k_1 + 1/k_2},$$

where, x_{eq} and k_{eq} are the equivalent displacement and stiffness of the series combination.

2-61

unstretched

stretched

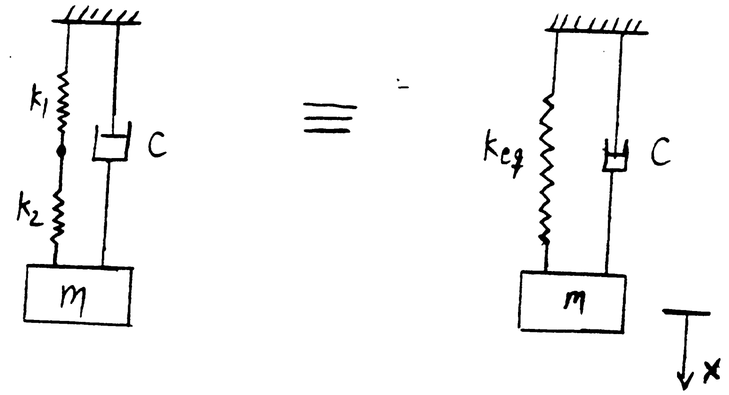


$$f = f_1 + f_2$$

$$\therefore k_{eq}x = k_1x + k_2x \Longrightarrow k_{eq} = k_1 + k_2,$$

where, k_{eq} is the equivalent stiffness of the parallel combination.

2.62



Equation of motion

$$m\ddot{x} = -k_{eq}x - c\dot{x} \implies m\ddot{x} + c\dot{x} + k_{eq}x = 0,$$

where, $k_{eq} = \dfrac{1}{1/k_1 + 1/k_2}$ (see problem 2.60) is the effective spring constant for the system.

3-1

$$x_{res.} = \frac{F}{c\,\omega_n} = \frac{F\tau}{2\pi c}$$

$$c = \frac{F\tau}{2\pi x_{res}} = \frac{24.46 \times 0.20}{2\pi \;\; 1.27\times 10^{-2}} = 61.3 \quad \frac{Ns}{m}$$

3-2

$$\frac{X_{undamped}}{X_{damped}} = \sqrt{\frac{(\omega_n^2-\omega^2)^2+(c\omega/m)^2}{(\omega_n^2-\omega^2)^2}} = R$$

$$\omega_n = \frac{2\pi}{\tau} = \frac{6.283}{.20} = 31.416 \qquad \omega = 8\pi = 25.13$$

$$\frac{c\omega}{m} = \frac{61.3 \times 8\pi}{1.95} = 790.1$$

$$R = \sqrt{\frac{(31.4^2-25.13^2)^2+(790.1)^2}{(31.4^2+25.13^2)^2}} = 2.44$$

3-3

$$\delta = \ln 4.2 = 1.435 = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

square & solve for ζ^2

$$2.0595(1-\zeta^2) = 39.478\,\zeta^2$$

$$\zeta^2 = \frac{2.059}{41.538} = .0496\,, \qquad \zeta = .223$$

$$\omega_d = \frac{2\pi}{\tau_d} = \frac{2\pi}{1.80} = \omega_n\sqrt{1-\zeta^2} \qquad \omega_n = \frac{2\pi}{1.80\sqrt{1-.0496}} = 3.5806$$

$$\omega = 3 \qquad \frac{\omega}{\omega_n} = .8378$$

Eq (3.1-7)
$$X = \frac{F_0/k}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\frac{\omega}{\omega_n}\right]^2}} = \frac{2/525}{\sqrt{[1-.702]^2+.1396}}$$

$$= \frac{2/525}{.4779} = .00797 \text{ m} = .797 \text{ cm}$$

Eq (3.1-8)

$$\phi = \tan^{-1}\frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)^2} = \tan^{-1}\frac{.446\times .8378}{.29801} = \tan^{-1}1.2538 = 51.43^\circ$$

$$m\ddot{x}_1 = -c\dot{x}_1 + k(x_2 - x_1)$$

$$m\ddot{x}_1 + c\dot{x}_1 + kx_1 = kX_2 \sin\omega t$$

Replace excitation by $kX_2 e^{i\omega t}$, then $x = X_1 e^{i(\omega t - \phi)}$

$$= X_1 e^{-i\phi} e^{i\omega t} = \bar{X}_1 e^{i\omega t}$$

$$\left[(k - m\omega^2) + i\omega c\right]\bar{X}_1 \cancel{e^{i\omega t}} = kX_2 \cancel{e^{i\omega t}}$$

$$\bar{X}_1 = \frac{kX_2}{(k - m\omega^2) + i\omega c} = \frac{kX_2 e^{-i\phi}}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

$$\therefore X_1 = \frac{kX_2}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}, \quad \phi = \tan^{-1}\frac{c\omega}{k - m\omega^2}$$

3-8

$$m\ddot{x} = c(\dot{y} - \dot{x}) + kx$$

$$m\ddot{x} + c\dot{x} + kx = c\dot{y}$$

let $y = Ye^{i\omega t}$ $\qquad x = Xe^{i(\omega t - \phi)} = Xe^{-i\phi}e^{i\omega t}$

$\dot{y} = i\omega Y e^{i\omega t}$ $\qquad = \bar{X}e^{i\omega t}$

$$(k - m\omega^2 + i\omega c)\bar{X} = i\omega Yc$$

$$\bar{X} = \frac{i\omega Yc}{k - m\omega^2 + i\omega c} = \frac{i\omega Yc e^{-i\psi}}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} = Xe^{-i\phi}$$

$$Xe^{-i\phi} = \frac{\omega Yc}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}\, i e^{-\psi} \qquad \therefore e^{-i\phi} = ie^{-i\psi} = e^{-i(\psi - \frac{\pi}{2})}$$

$$X = \frac{\omega Yc}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \qquad \phi = \psi - \frac{\pi}{2}, \quad \tan\psi = \frac{c\omega}{k - m\omega^2}$$

3-4 Square Eq. (3.1-7)

$$\left(\frac{X}{X_0}\right)^2 = \frac{1}{(1-r)^2+4\zeta^2 r} \qquad \text{where } X_0 = F/k, \quad r = (\omega/\omega_n)^2$$

$$\frac{\partial}{\partial r}\left(\frac{X}{X_0}\right)^2 = \frac{2(1-r)-4\zeta^2}{(\text{denom.})^2} = 0 \quad \therefore r = 1-2\zeta^2 = (\omega/\omega_n)_p^2$$

$$\left(\frac{\omega}{\omega_n}\right)_p = \sqrt{1-2\zeta^2}$$

3-5 At resonance $\frac{\omega}{\omega_n} = 1.0$, $\quad \frac{X}{X_0} = \frac{1}{2\zeta} = \frac{.58}{X_0}$

When $\frac{\omega}{\omega_n} \neq 1.0$ $\qquad \therefore X_0 = 1.16\zeta$

$$\frac{X}{X_0} = \frac{1}{\sqrt{[1-r^2]^2+[2\zeta r]^2}} = \frac{.46}{X_0}$$

Square & solve for ζ^2

$$\frac{1}{[1-.64]^2+[2\zeta]^2 .64} = \frac{.2116}{(1.16\zeta)^2}$$

$$6.359\zeta^2 = .1296 + 2.560\zeta^2 \qquad \zeta^2 = .0341$$
$$\zeta = .1847$$

3-6

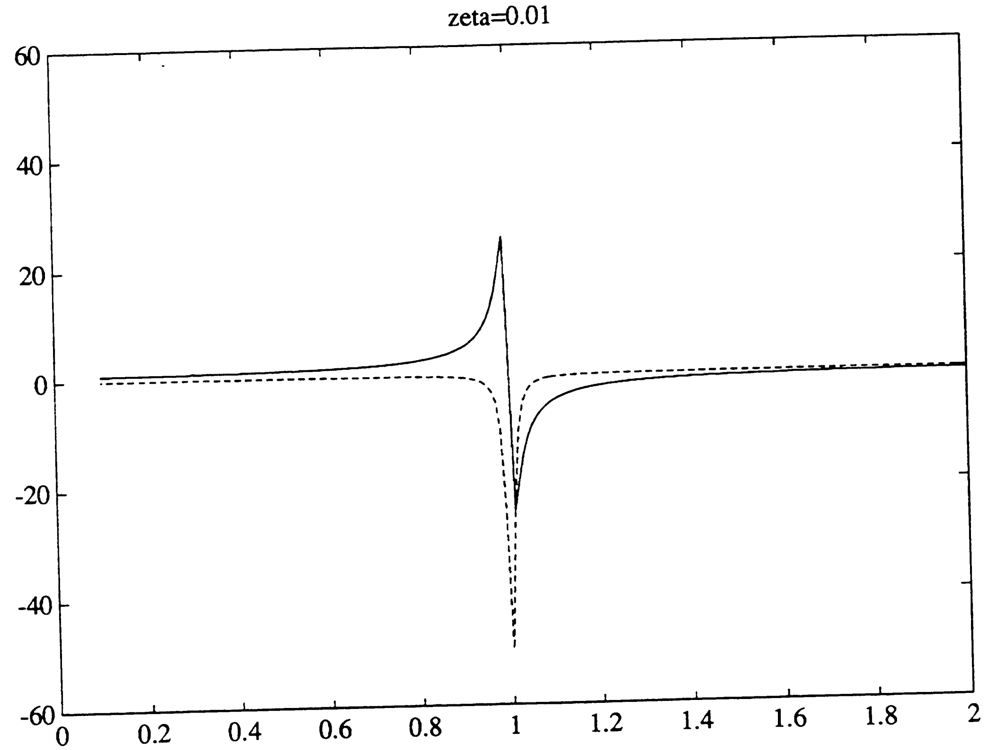
Eq. 3.1-17 $\quad H(\omega) = \frac{1-r^2}{[1-r^2]^2+[2\zeta r]^2} - i\frac{2\zeta r}{[1-r^2]^2+[2\zeta r]^2}$

$$= R + iI$$

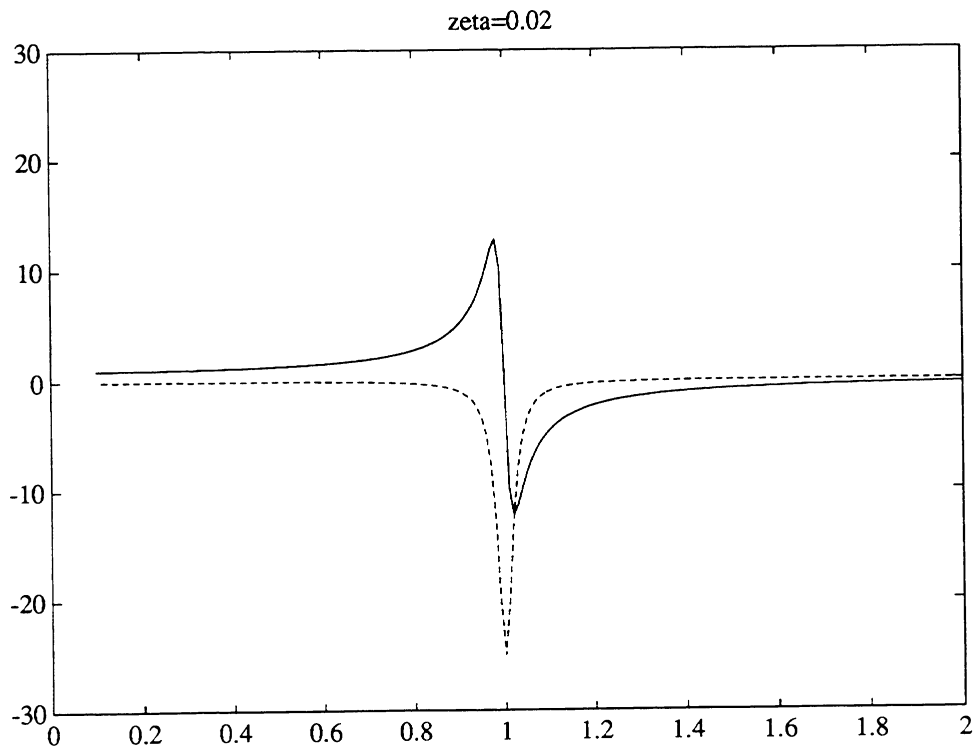
```
%This is the function file for the imaginary part of prob. 3-6
function [imagp]=imp(r)
zeta=0.01;
imagp=-(2*zeta.*r)./((1-r.*r).^2+(2*zeta.*r).^2);
end
```

```
%This is the function file for the real part of prob. 3-6
function [realp]=rp(r)
zeta=.01;
realp=(1-r.*r)./((1-r.*r).^2+(2*zeta.*r).^2);
end
```

prob 3-6 cont

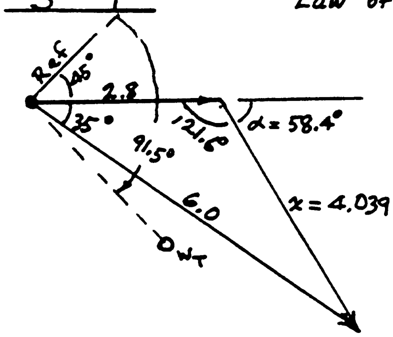


zeta=0.01
60
40
20
0
-20
-40
-60
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2



zeta=0.02
30
20
10
0
-10
-20
-30
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

3 − 9



Law of Cosines

$$x^2 = 2.8^2 + 6.0^2 - 2 \times 2.8 \times 6.0 \times \cos 35^\circ$$

$$= 7.84 + 36.0 - 33.60 \times .8192 = 16.316$$

$$x = 4.039$$

Law of Sines

$$\frac{4.039}{\sin 35^\circ} = \frac{6.0}{\sin \alpha}$$

$$\sin \alpha = \frac{6}{4.039} \times .5736 = .8521$$

$$\alpha = 58.44^\circ$$

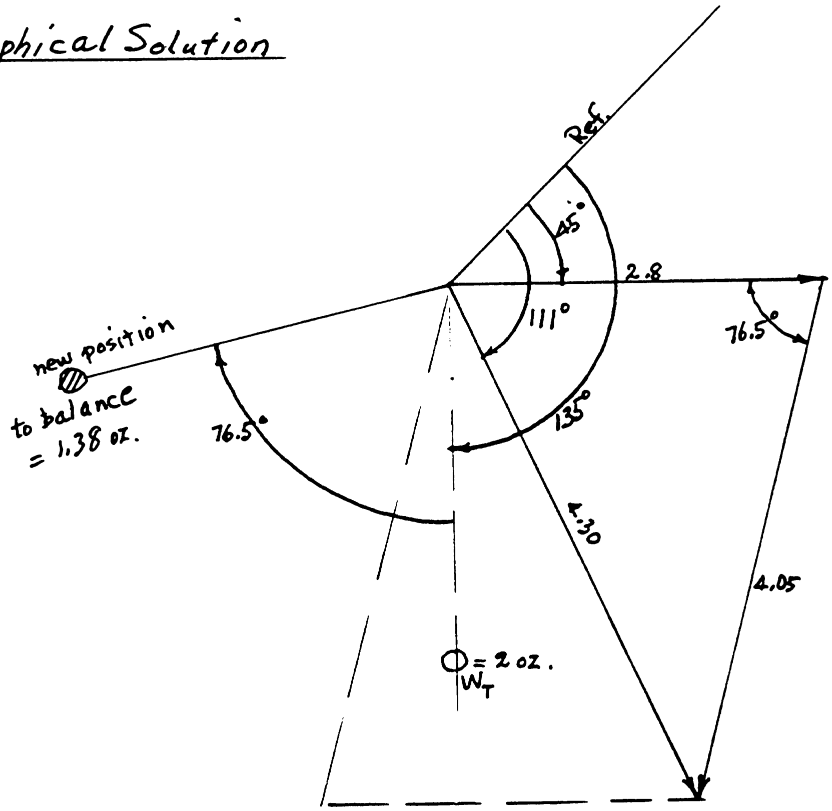
$$180 - \alpha = 121.6^\circ$$

$$2 \text{ oz.} \times \frac{2.8}{4.036} = 1.386 \text{ oz.}$$

To balance, rotate the vector $x = 4.039$, 121.6° clockwise and multiply by $\frac{2.8}{4.036}$. The position of the balancing wt is found by rotating the trial weight 121.6° c.w. from the position (of 91.5° from ref.) and reducing its value to 1.386 oz.

3-10

Graphical Solution



Original unbalance 2.8 @ 45 c.w. from Ref.

With trial weight $W_T = 2$ @ 135° c.w. " " new reading = 4.30 @ 111° c.w.

Resultant = 4.05 @ 76.5° from horizontal

To balance rotate W_T 76.5° and reduce to $\frac{2.8}{4.05} \times 2 = 1.38$ oz.

Students should check with prob. 3-9

3-11

From Eq. 3.2-5, we have

$$\tan\phi = \frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)^2} = \frac{2\times 0.10\times\frac{6}{9}}{1-\left(\frac{6}{9}\right)^2} = \frac{0.1333}{1-0.444}$$

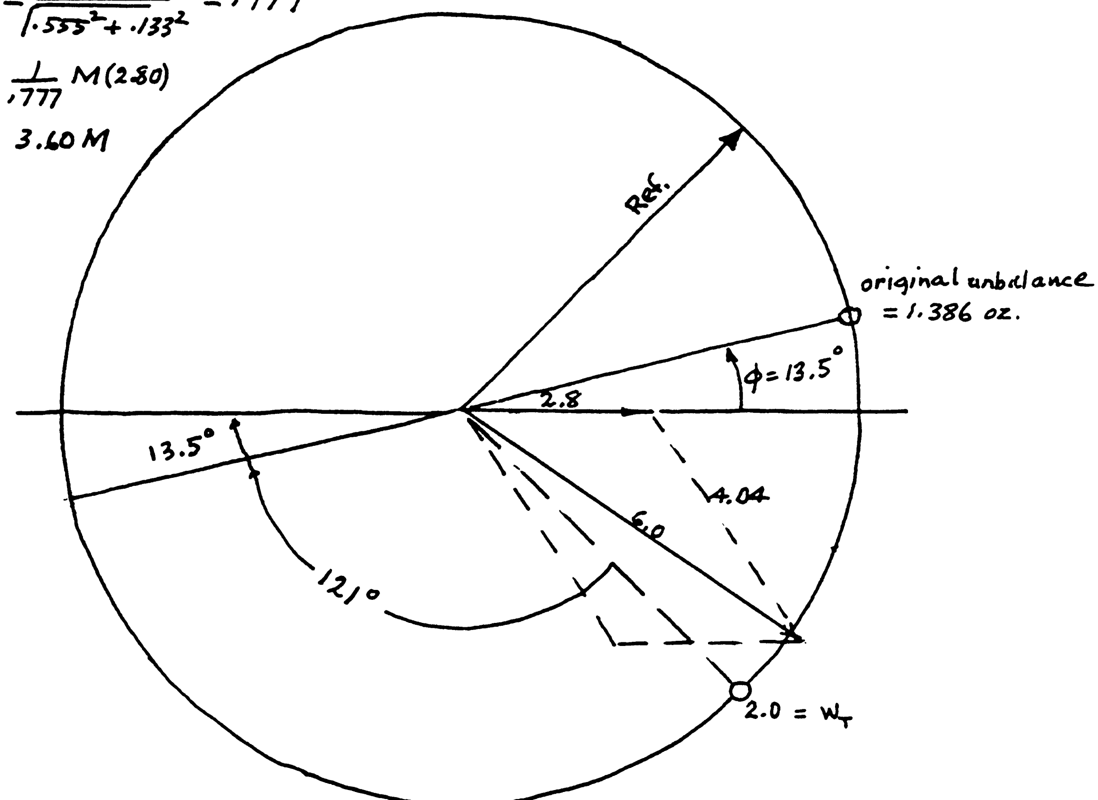
$$= 0.2399 \qquad \therefore \underline{\underline{\phi = 13.49^\circ}}$$

Eq. 3.2-4

$$\frac{MX}{me} = \frac{.444}{\sqrt{.555^2+.133^2}} = .777$$

$$me = \frac{1}{.777}M(2.80)$$

$$= 3.60\,M$$



Original unbalance $= 2.8 = X$

$MX = .777\,me \quad \therefore\ (me)_1 = 3.60\,M$

2 oz trial wt W_T resulted in $X = 4.04$

$M\,4.04 = .777(me)_2 \quad \therefore\ (me)_2 = 5.199\,M$

$$\frac{(me)_2}{(me)_1} = \frac{5.199}{3.60} = 1.444 \qquad \therefore\ (me)_2 = 1.444\times 1.386 = 2.0 \text{ oz}$$

$$= W_T$$

3-12 The method used to balance the single disk can be modified to balance the long rotor ahown in Fig P3-12 (a).

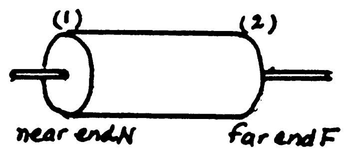


Fig P3-12 (a)

(1) First measure the original unbalance vectors w.r.t. a fixed reference on the rotor at the near and far ends as N and F respectively.

(2) Next, a trial weight W_{t1} is added to the near end and the new vectors N_1 and F_1 , shown in Fig P3-12 (b) are recorded with W_{t1} . The dotted vectors A and αA are due to W_{t1} alone,

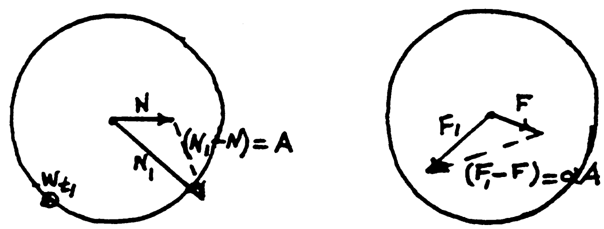


Fig P3-12 (b)

and since they are proportional to W_{t1} , they can be related to each other by a complex influence coefficient, or operator α .

(3) Next, the trial weight W_{t1} is removed and a new trial weight W_{t2} is added to the far end. Measurements for N_2 and F_2 are then recorded, as shown in Fig P3-12 (c).

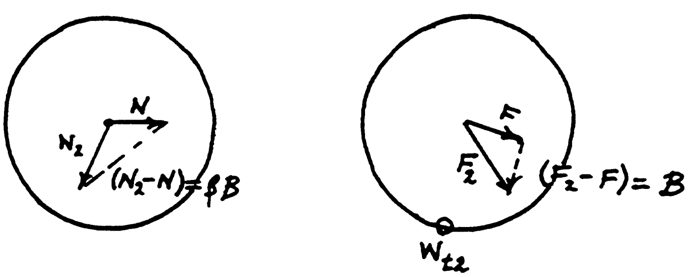


Fig P 3-12 (c)

3-12 Cont.

Again, the dotted vectors are proportional to W_{t2} , and they are related to each other by another complex operator β.

(4) As in the single disk problem, the two dotted vectors for the near end must be rotated and combined to cancel the original unbalance N. Also the final balance weights W_{b1} and W_{b2} must be related to the trial weights W_{t1} and W_{t2} by two additional complex operators θ and ϕ .

$$W_{b1} = \theta W_{t1}$$
$$W_{b2} = \phi W_{t2}$$

Thus to balance the long rotor, the following two equations must be satisfied

$$\theta A + \phi(\beta B) = -N$$
$$\phi B + \theta(\alpha A) = -F$$

Solving these for θ and ϕ , we obtain

$$\theta = \frac{\beta F - N}{(1-\alpha\beta) A}$$

$$\phi = \frac{\alpha N - F}{(1-\alpha\beta) B}$$

In these equations, N and F are the original unbalance vectors, and α and β are obtained by the division $\alpha = \frac{\alpha A}{A}$ and $\beta = \frac{\beta B}{B}$. With θ and ϕ known, the balancing weights W_{b1} and W_{b2} are found.

3 - 13

(a) $\varphi = 90^\circ$ ∴ resonance $\quad \omega_n = \omega = \frac{900 \times 2\pi}{60} = 2\pi f_n$

$f_n = 15\ cps = 900\ cpm$

(b) Eq. (3.2-4) at resonance

$$\zeta = \frac{em}{2MX} = \frac{2 \times .0921}{2 \times 181.4 \times 21.6 \times 10^{-3}} = 0.0118 \times 2$$

(c) At 1200 rpm $\quad \frac{\omega}{\omega_n} = \frac{1200}{900} = 1.333$

Eq. (3.2-4)

$$X = \frac{me}{M} \frac{1.333^2 \times 2}{\sqrt{[1 - 1.333^2]^2 + [2 \times .0118 \times 1.333]^2}}$$

$$= \frac{2 \times .0921}{181.4} \times \frac{1.777}{.78023} = .002314\,m = .2314\,cm$$

(d)

Eq. (3.2-5) $\quad \varphi = \tan^{-1} \frac{2 \times .0235 \times 1.333}{1 - 1.333^2} = \tan^{-1}(-.0805)$

$$= 180^\circ - 4.61^\circ = 175.4^\circ$$

3 - 14

$$M\ddot{x} + c\dot{x} + kx = (me\omega^2)\sin\omega t$$

Let $(me\omega^2)\sin\omega t = F e^{i\omega t}$

then $x = X e^{i(\omega t - \varphi)} = X e^{-i\varphi} e^{i\omega t} = \bar{X} e^{i\omega t}$

$$(-\omega^2 M + ic\omega + k)\bar{X} \cancel{e^{i\omega t}} = F \cancel{e^{i\omega t}}$$

$$\bar{X} = \frac{F}{(k - \omega^2 M) + i(c\omega)}$$

3 – 15

$$\omega = \frac{1200}{60} = 20 \text{ rps} = 20 \times 2\pi \text{ rad/s.}$$

$$\omega_n = 18 \text{ cps} = 18 \times 2\pi \text{ rad/s} \quad \therefore \frac{\omega}{\omega_n} = 1.111$$

Eq. 3.2-6

$$x(t) = \underbrace{X_1 e^{-\zeta\omega_n t} \sin(\sqrt{1-\zeta^2}\,\omega_n t - \phi_1)}_{\text{transient}} + \underbrace{\frac{me\omega^2 \sin(\omega t - \varphi)}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}}_{\text{steady state}}$$

at $t = 0$, $x(0) = 0$

$$0 = -X_1 \sin\phi_1 + \frac{(me\omega^2/k)\sin(-\varphi)}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\zeta\,\omega/\omega_n]^2}} \qquad (1)$$

at $t=0$, $\dot{x}(0) = 0$

$$\dot{x}(t) = X_1 e^{-\zeta\omega_n t}\left[\sqrt{1-\zeta^2}\,\omega_n \cos(\sqrt{1-\zeta^2}\,\omega_n t - \phi_1) - \zeta\omega_n \sin(\sqrt{1-\zeta^2}\,\omega_n t - \phi_1)\right]$$
$$+ \frac{(me\omega^2/k)\,\omega \cos(\omega t - \varphi)}{\sqrt{[1-(\frac{\omega}{\omega_n})^2]^2 + [2\zeta \frac{\omega}{\omega_n}]^2}}$$

$$\therefore \dot{x}(0) = 0 = X_1\left[\omega_n\sqrt{1-\zeta^2}\cos\phi_1 + \zeta\omega_n \sin\phi_1\right] + \frac{(me\omega^2/k)\,\omega\cos\varphi}{\sqrt{\quad}} \qquad (2)$$

from (1) $\quad X_1 \sin\phi_1 = \dfrac{-(me\omega^2/k)\sin\varphi}{\sqrt{\quad}}$

from (2)

$$X_1\omega_n\sqrt{1-\zeta^2}\cos\phi_1 + \zeta\omega_n\left[\frac{-(me\omega^2/k)\sin\varphi}{\sqrt{\quad}} + \frac{(me\omega^2/k)\,\omega\cos\varphi}{\sqrt{\quad}}\right] = 0$$

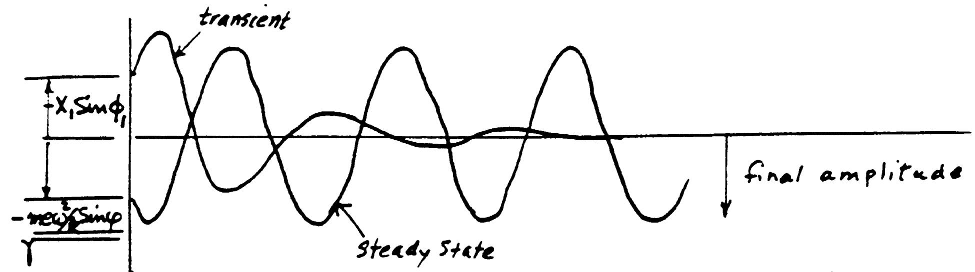
dividing

$$\therefore \tan\phi_1 = \frac{-\sqrt{1-\zeta^2}\,\sin\varphi}{\zeta\sin\varphi - \frac{\omega}{\omega_n}\cos\varphi} \qquad (3)$$

3-15 Cont.

at $t = \infty$, transient term $= 0$ due to $e^{-\zeta\omega_n t} \to 0$

$$\therefore \text{final ampl.} = x(\infty) = \frac{(me\omega^2/k)\sin(\omega t - \varphi)}{\sqrt{[1-(\frac{\omega}{\omega_n})^2]^2 + [2\zeta\frac{\omega}{\omega_n}]^2}}$$



phase φ can then be solved from $\tan\varphi = \dfrac{2\zeta\frac{\omega}{\omega_n}}{1-(\frac{\omega}{\omega_n})^2} = \dfrac{.2(1.111)}{1-(1.111)^2}$

$= -.94755 \quad \therefore \varphi = -43.457°$ Eq. (3.2-5)

Then ϕ_1 solved from (3) $\tan\phi_1 = -.7820$, $\phi_1 \doteq 142°$

Then solve for X_1 and sub. into Eq. (3.2-6) for build up eq.

3-16 Lowest critical speed = fundamental freq. of lateral vibr.

$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ where $k = \frac{48EI}{\ell^3}$, $I = \frac{\pi d^4}{64}$

$m = m_{disk} + 0.486\, m_{shaft}$

$$k = \frac{48(29\times10^6)\pi(\frac{1}{2})^4}{(24)^3 \times 64} = 309. \text{ lb/in}$$

$$.486\, m_{shaft} = \frac{.486(.283)\frac{\pi}{4}(\frac{1}{2})^2(24)}{386} = \frac{.648}{386}$$

$$m = \frac{10}{386} + \frac{.648}{386} = \frac{10.65}{386}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{309\times386}{10.65}} = 16.84 \text{ cps} = 1028 \text{ rpm}$$

3-17

$w = 10 \text{ lb} = 44.48 \text{ N} \qquad m = \frac{44.48}{9.81} = 4.534 \text{ kg}$

$g = 386 \frac{in}{sec^2} = 9.81 \ m/s^2$

$$k = \frac{48EI}{\ell^3} = \left\{ \begin{array}{l} E = 200 \times 10^9 \ N/m^2 \\ \ell = 2 \times .3048 = .6096 \ m \\ d = .5 \times 2.54 \times 10^{-2} = 1.270 \times 10^{-2} \ m \\ I = \frac{\pi d^4}{64} = .1277 \times 10^{-8} \end{array} \right.$$

$$k = \frac{200 \times 10^9 \times 48 \times .1277 \times 10^{-8}}{(.6096)^3} = 54116 \ N/m$$

$$.486 \, m_{shaft} = .486 \times \frac{\pi}{4}(1.270 \times 10^{-2})^2 (.6096) \times \rho = .2938$$

$\rho = 7830 \frac{kg}{m^3}$ = density of steel

$$f = \frac{1}{2\pi}\sqrt{\frac{54116}{4.534 + .2938}} = 16.86 \text{ Hz}$$

3-18

dia $= 2.54$ cm $\qquad I = \frac{\pi d^4}{64} = \frac{\pi \times 41.62}{64 \times 100^4} = 2.043 \times 10^{-8} \ m^4$

$$k = \frac{48EI}{\ell^3} = \frac{48(200 \times 10^9) 2.043 \times 10^{-8}}{\mathbf{.4064^2}} = 2.922 \times 10^6 \ N/m$$

$$m = 13.6 + .486\left(\frac{\pi}{4} \times .0254^2 \times .4064\right)(7830) = 13.6 + .784$$
$$= 14.38 \text{ kg}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{2.922 \times 10^6}{14.38}} = 71.74 \text{ Hz} = 4304 \text{ rpm}$$

$$\frac{\omega}{\omega_n} = \frac{6000}{4304} = 1.394 \qquad r = \frac{e(\frac{\omega}{\omega_n})^2}{1 - (\frac{\omega}{\omega_n})^2} = -2.060 e$$

$me = .2879$ kg cm (given) $\qquad \therefore \quad e = \frac{.2879}{13.6} = .02117$ cm

$r = -\mathbf{2060}(.02117) = -.04316$ cm

$$F = m(r+e)\omega^2 = 14.38\left(\frac{-.04316 + .02117}{100}\right)(2\pi \times 100)^2 = 1273 \text{ N}$$

3-18 Cont.

for diam $= 1.905$ cm

$$.486\, m_{shaft} = .486\left(\frac{\pi}{4} \times .01905^2 \times .4064\right)(7830) = .4408$$

$$m = m_{disk} + .483\, m_{shaft} = 14.04 \text{ kg.}$$

$$I = \frac{\pi}{64}\, 1.905^4 \times 100^{-4} = .6464 \times 10^{-8}$$

$$k = \frac{48\,(200 \times 10^9)\,.6464 \times 10^{-8}}{.4064^3} = 0.9441 \times 10^6$$

$$f_n = 41.27 \text{ Hz} = 2476 \text{ rpm} \qquad \frac{\omega}{\omega_n} = \frac{6000}{2476} = 2.423$$

$$r = \frac{.02117\,(2.423)^2}{1 - (2.423)^2} = -.02552 \qquad r + e = .00435$$

$$F = 14.042\left(\frac{.00435}{100}\right)\left(2\pi \times \frac{6000}{60}\right)^2 = 241.1 \text{ N}$$

3-19 $\quad r = r_0 + \dfrac{e\omega t}{2}$ (see Ex. 3.4-1)

$$.0508 = 0 + .0212\,(2\pi \times 100)\frac{t}{2}$$

$$t = \frac{.0508}{6.6602} = .0075 \text{ sec}$$

3-20 $\quad m\ddot{x} = -k(x - y)$ where x = displ. of m measured from static equilib. position of m with $y = 0$

Let $y = Y \sin \dfrac{2\pi V t}{L}$

then $m\ddot{x} + kx = kY \sin \dfrac{2\pi V t}{L} = kY \sin \omega t$

where $\omega = \dfrac{2\pi V}{L}$ $\qquad$ Sol is $x = X \sin \omega t$

$$X = \frac{Y}{1 - (\omega/\omega_n)^2} \qquad \omega_n = \sqrt{\frac{k}{m}}$$

The most unfavorable speed corresponds to $\dfrac{\omega}{\omega_n} = 1$

$$\therefore\ V = \frac{L}{2\pi}\sqrt{\frac{k}{m}}$$

3 – 21 (Ref. Prob 3–20)

Eq. (2.2–9)
$$f_n = \frac{15.76}{\sqrt{\Delta_{mm}}} = \frac{15.76}{\sqrt{101.6}} = 1.563 \text{ Hz}$$

$$\omega_n = 2\pi f_n = 9.824 \text{ r/s}$$

$$V_{crit.} = \frac{L\omega_n}{2\pi} = \frac{14.63}{2\pi} \times 9.82 = 22.87 \text{ m/s.}$$

$$V \quad 64.4 \text{ km/hr} = \frac{64400}{60^2} = 17.89 \text{ m/s}$$

$$\omega = \frac{2\pi V}{L} = \frac{2\pi\ 17.89}{14.63} = 7.683 \text{ r/s}$$

$$\left(\frac{\omega}{\omega_n}\right)^2 = \left(\frac{7.683}{9.824}\right)^2 = 0.6117 \qquad X = \frac{7.62 \text{ cm}}{1 - .6117} = 19.62 \text{ cm}$$

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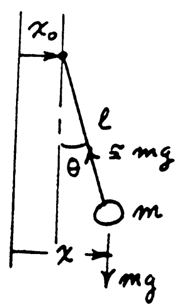
3 – 22



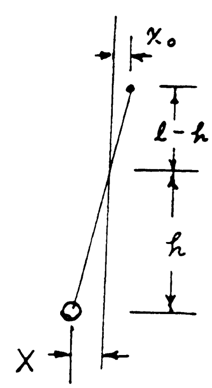
$$x \cong x_o + l\theta \qquad \theta = \frac{x - x_o}{l}$$

$$m\ddot{x} = -mg\theta = -\frac{mg}{l}(x - x_o)$$

$$\ddot{x} + \frac{g}{l}x = \frac{g}{l}x_o$$

Let $x_o = X_o \sin \omega t \qquad x = X \sin \omega t$

$$\therefore X = \frac{X_o}{1 - (\omega/\omega_n)^2} \qquad \omega_n = \sqrt{\frac{g}{l}}$$



when $\omega = \sqrt{2}\,\omega_n \qquad X = -X_o$

$\therefore$ mode found at $\frac{l}{2}$

$$\frac{h}{|X|} = \frac{l-h}{|X_o|}$$

$$h\left|\frac{X_o}{X}\right| = l - h = h\left[\left(\frac{\omega}{\omega_n}\right)^2 - 1\right] \qquad \text{since } \frac{\omega}{\omega_n} > 1$$

$$h = l\left(\frac{\omega_n}{\omega}\right)^2$$

3-23 $\quad m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$

$y = Y\sin\omega t \qquad x = X\sin(\omega t - \psi) \qquad \dot{x} = \omega X\cos(\omega t - \psi)$

$\dot{y} = \omega Y\cos\omega t \qquad \ddot{x} = -\omega^2 X\sin(\omega t - \psi)$

Sub. into DE.

$(k - m\omega^2)X\sin(\omega t - \psi) + c\omega X\cos(\omega t - \psi) = c\omega Y\cos\omega t + kY\sin\omega t$

Expand $\sin(\omega t - \psi)$ & $\cos(\omega t - \psi)$ & equate coef of $\cos\omega t$ & $\sin\omega t$

coef. of $\sin\omega t \rightarrow \quad [(k - m\omega^2)\cos\psi + \omega c\sin\psi]X = kY$

" " $\cos\omega t \rightarrow \quad [(k - m\omega^2)\sin\psi - \omega c\cos\psi]X = -\omega cY$

Divide & factor $\cos\psi$ from num. & denom. to get

$$\tan\psi = \frac{mc\omega^3}{k(k - m\omega^2) + (c\omega)^2} = \text{Eq.}(3.5\text{-}9)$$

Solve for $\sin\psi$ & $\cos\psi$ & sub. into

$$\frac{X}{Y} = \frac{k}{(k - m\omega^2)\cos\psi + \omega c\sin\psi} = \sqrt{\frac{k^2 + (\omega c)^2}{(k - m\omega^2)^2 + (c\omega)^2}} = \text{Eq.}(3.5\text{-}8)$$

after much algebra

3-24

$$f_{cpm} = 188\sqrt{\frac{1}{\Delta''}\left(\frac{1}{TR} + 1\right)} \qquad \text{(See Sec. 3.6)}$$

$$1600_{cpm} = 188\sqrt{\frac{1}{\Delta''}\left(\frac{1}{.15} + 1\right)}, \quad \therefore \Delta'' = .106''$$

25.4 mm = 1 in

$\therefore \Delta_{mm} = 2.69$ mm.

For $f = 2200$ cpm the F_{TR} is smaller

3-25 $\quad TR = \dfrac{1}{\left(\frac{\omega}{\omega_n}\right)^2 - 1} = 0.10 \qquad \therefore \left(\dfrac{\omega}{\omega_n}\right)^2 = 11.0$

$$\omega_n^2 = \frac{k}{m} = \frac{\omega^2}{11.0} \qquad k = \frac{m\omega^2}{11} = \frac{65}{386}\left(\frac{580 \times 2\pi}{60}\right)^2\frac{1}{11} = 56.5 \text{ lb/in}$$

$$k_{per\ spring} = \frac{1}{3} \times 56.5 = 18.8 \text{ lb/in}$$

3-26

$$k = \frac{Mg}{\Delta} = \frac{453.4 \times 9.81}{.005080} = 875561 \ \frac{N}{m} = 8755.6 \ \frac{N}{cm}$$

$$\omega_n^2 = \frac{k}{M} = \frac{8755.6 \times 10^2}{453.4} = 1931.1$$

$$\left(\frac{\omega}{\omega_n}\right)^2 = \left(\frac{1200 \times 2\pi}{60}\right)^2 \frac{1}{1931.1} = 8.177$$

$$X = \frac{\frac{me}{M}\left(\frac{\omega}{\omega_n}\right)^2}{\left(\frac{\omega}{\omega_n}\right)^2 - 1} = \frac{\frac{.2303}{453.4} \times 8.177}{8.177 - 1} = 0.000579 \ m$$

$$F_{TR} = kX = 875561 \times .000579 = 506.7 \ N$$

3-27

New $M = 453.4 + 1136. = 1589.4$ kg

New $k = 875561 \times \frac{1589.4}{453.4} = 3069.295 \times 10^3 \ \frac{N}{m}$

$\frac{\omega}{\omega_n}$ is same

$$X = \frac{me}{M} \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\left(\frac{\omega}{\omega_n}\right)^2 - 1} = \frac{453.4}{1589.4} \times .000579 = 0.165 \times 10^{-3} \ m$$

3-28

$M = 68 + 1200 = 1268$ kg.

$f_n = 160$ cpm $\quad \omega_n = \frac{160}{60} \times 2\pi = 16.75$ r/s

$\frac{\omega}{\omega_n} = \frac{31.4}{16.75} = 1.8746 \qquad k = \omega_n^2 M = 355951 \ \frac{N}{m}$

$$X = \frac{100 / 0.3559 \times 10^6}{\sqrt{[1 - 1.875^2]^2 + [0.2 \times 1.875]^2}} = 110.5 \times 10^{-6} \ m$$

$$= 0.01105 \ cm$$

$$F_{TR} = kX\sqrt{1 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2} = .3559 \times 10^6 \times 110.5 \times 10^{-6} \sqrt{1 + .1406}$$

$$= 42.0 \ N$$

3-29

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{280200}{113}} = 49.796$$

$$\omega = 2\pi \times 20 = 125.6 \qquad \frac{\omega}{\omega_n} = 2.5236$$

$$\text{Accel.} = 15.24 = \omega^2 Y \qquad \therefore \quad Y = .000965 \text{ cm}$$

$$\left|\frac{X}{Y}\right| = \frac{\sqrt{1 + [.2 \times 2.5236]^2}}{\sqrt{[1 - 6.368]^2 + .2547}} = 0.2078$$

$$\therefore X = .2078 \times .000965 = 0.0002005 \text{ cm}$$

$$\omega^2 X = 3.166 \text{ cm/s}^2 = \text{transmitted accel.}$$

3-30 Add mass M to instrument $\therefore$ increase ω/ω_n

$$X \text{ must be reduced to } \left(\frac{2.03}{3.166}\right) .0002005 = .0001285$$

$$\frac{X}{Y} = \frac{1285}{9650} = .1332 \le \sqrt{\frac{1 + (.2\frac{\omega}{\omega_n})^2}{[1 - (\frac{\omega}{\omega_n})^2]^2 + (.2\frac{\omega}{\omega_n})^2}}$$

Solve by trial: for $\frac{\omega}{\omega_n} = 3$, $\sqrt{\quad} = .1321$

for $\frac{\omega}{\omega_n} = 4$ $\sqrt{\quad} = .0893$

both of these are OK

$$\text{For } \frac{\omega}{\omega_n} = 4 \qquad \omega_n = 31.4 = \sqrt{\frac{k}{M_i + M_o}} = \sqrt{\frac{280200}{M_i + M_o}}$$

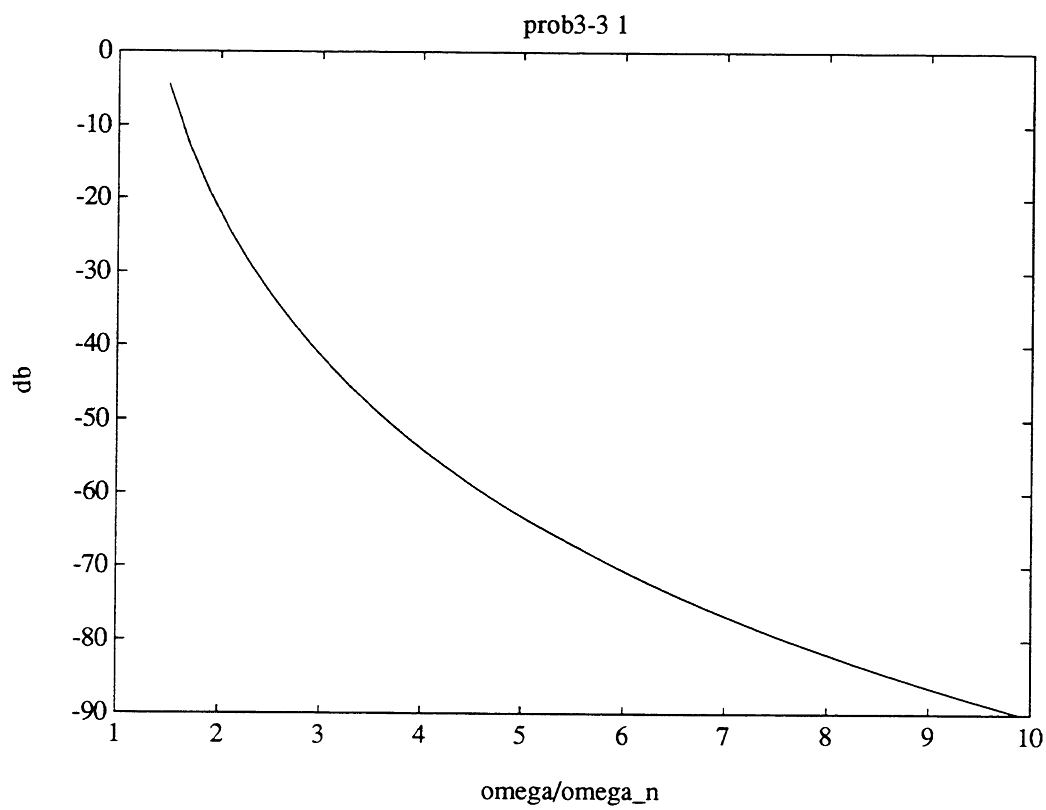
$$M_i + M_o = 284. \quad \therefore \quad M_o = 171. \text{ kg to be added.}$$

3-31 Compare Eq.(3.5-8) with Eq(3.6-2). They are same

$$TR = \sqrt{\frac{1+(2\zeta\frac{\omega}{\omega_n})^2}{[1-(\frac{\omega}{\omega_n})^2]^2+[2\zeta\frac{\omega}{\omega_n}]^2}}$$

To calculate $20 \log|TR|$, first calculate TR with ζ fixed, varying $\frac{\omega}{\omega_m}$. Then find Db.

```
%This is the code for prob. 3-31
ome=1.5:.2:10;
zeta=.02;
top=1+(2*zeta.*ome).^2;
bot=(1-ome.^2).^2+(2*zeta.*ome).^2;
tr=sqrt(top./bot)
db=20*log(abs(tr));
```



3-32 $W_d = \pi c \omega X^2 = \pi 2\zeta \frac{\omega}{\omega_n} k X^2$

$$= \frac{\pi F_o^2}{k} \frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2}$$

3-33

$$\eta = \frac{W_d}{2\pi U}, \qquad U = \frac{1}{2}kX^2_{max}, \qquad W_d = \pi c \omega X^2_{max}$$

$$\therefore \eta = \frac{\pi c \omega X^2}{2\pi \frac{1}{2} k X^2} = \frac{c\omega}{k}$$

3-34 $\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = \frac{F}{m}\sin\omega t$

$X_{res.} = \frac{F}{2\zeta k}$ From Prob. 3-29 $\eta_{res} = \frac{c\omega}{k} = \frac{c\omega_n}{k}$ at resonan.

$$= \frac{c}{c_{cr}} \frac{2m\omega_n^2}{k} = 2\zeta$$

$$\therefore \quad \ddot{x} + \eta_{res.}\omega_n\dot{x} + \omega_n^2 x = \frac{F}{m}\sin\omega t$$

3-35 $\frac{\tau_d}{\tau_n} = \frac{\omega_n}{\omega_d} = \frac{1}{\sqrt{1-\zeta^2}} \qquad \therefore \left(\frac{\tau_n}{\tau_d}\right)^2 = 1 - \zeta^2$

$\left(\frac{\tau_n}{\tau_d}\right)^2 + \zeta^2 = 1$ :: circle of radius 1

3-36 $\frac{W_d}{U} = \frac{2\zeta\pi k X^2}{\frac{1}{2}kX^2} = 4\zeta\pi$

but $\delta \cong 2\pi\zeta \quad \therefore \quad \frac{W_d}{U} \cong 2\delta, \qquad \zeta = \frac{c}{2m\omega_n}$

$$\delta = \frac{c\pi}{m\omega_n} = \frac{c\pi\omega_n}{k} = \frac{W_d}{2U}$$

3-37 $\quad W_d = f(X,\omega) = \pi c\omega X^2$

$$\delta = \frac{W_d}{2U} = \frac{\pi c\omega X^2}{kX^2} = \frac{\pi c\omega}{k} \quad \text{for viscous damping}$$

3-38 $\quad W_d = c_{eq.}\pi\omega X^2 = D\,4X$ for Coulomb

$$\therefore c_{eq.} = \frac{4D}{\pi\omega X} \qquad \zeta_{eq} = \frac{c_{eq}}{c_c} = \frac{4D}{\pi\omega X\, 2m\omega_n}$$

$$2\zeta_{eq}\frac{\omega}{\omega_n} = \frac{4D}{\pi\omega X\, 2m\omega_n}\cdot\frac{2\omega}{\omega_n} = \frac{4D}{\pi kX}$$

3-39

$$X = \frac{F_0/k}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta_{eq.}\frac{\omega}{\omega_n}\right]^2}} \qquad 2\zeta_{eq.}\frac{\omega}{\omega_n} = \frac{4D}{\pi kX}$$

Square both sides and subst for $\zeta_{eq.}$

$$X^2\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[\frac{4D}{\pi kX}\right]^2\right\} = (F_0/k)^2$$

$$X^2\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 = \left(\frac{F_0}{k}\right)^2-\left(\frac{4D}{\pi k}\right)^2$$

$$X = \frac{\sqrt{(F_0/k)^2-(4D/\pi k)^2}}{1-\left(\frac{\omega}{\omega_n}\right)^2} \quad \therefore \text{for } X \text{ real} \quad F_0 > \frac{4D}{\pi}$$

3-40 Rewrite eq. as

$$\frac{kX}{F_0} = \frac{\sqrt{1-\left(\frac{4}{\pi}\frac{D}{F_0}\right)^2}}{1-\left(\frac{\omega}{\omega_n}\right)^2} = f\left(\frac{D}{F_0},\frac{\omega}{\omega_n}\right)$$

3-41 $\quad J\ddot{\theta}_2 + K(\theta_2-\theta_1) = 0 \qquad \omega_n^2 = \frac{K}{J}$

$$\ddot{\theta}_2 + \omega_n^2\theta_2 = \omega_n^2\theta_1 = \omega_n^2\Theta_1\sin\omega t \qquad \text{Let } \theta_2 = \Theta_2\sin\omega t$$

$$\therefore \Theta_2 = \frac{\omega_n^2\Theta_1}{\omega_n^2-\omega^2} \quad \text{and} \quad \theta_2-\theta_1 = (\Theta_2-\Theta_1)\sin\omega t \qquad \begin{cases}\theta_2 = \text{outer wheel}\\ \theta_1 = \text{shaft.}\end{cases}$$

$$\text{Rel. ampl} = (\Theta_2-\Theta_1) = \frac{\omega_n^2\Theta_1}{\omega_n^2-\omega^2}-\Theta_1 = \frac{\omega^2}{\omega_n^2-\omega^2}\Theta_1$$

$$= \frac{\left(\frac{\omega}{\omega_n}\right)^2}{1-\left(\frac{\omega}{\omega_n}\right)^2}\Theta_1$$

$$\text{Rel to fixed ref} \quad \Theta_2 = \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2}\Theta_1$$

3-42 Let $r = \omega/\omega_n$

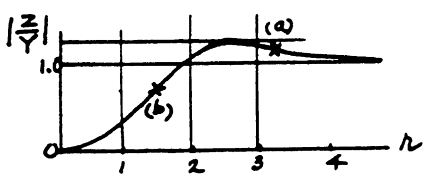
$$\frac{Z}{Y} = \frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \qquad \left(\frac{Z}{Y}\right)^2 = \frac{(r^2)^2}{(1-r^2)^2+(2\zeta r)^2}$$

$$\frac{\partial}{\partial r^2}\left(\frac{Z}{Y}\right) = 0 \quad \text{gives} \quad r_p^2 = \frac{1}{1-2\zeta^2} \quad \text{for peak ampl.}$$

for $\zeta = 0.650$, $r_p = 2.54$, $\frac{Z}{Y} = 1.012$

for (a) $\frac{Z}{Y} = 1.010$ Write above eq. as $r^4\left[1-\left(\frac{Y}{Z}\right)^2\right] - r^2(2-4\zeta^2)+1 = 0$

For accuracy expand $1-\left(\frac{Y}{Z}\right)^2$



$$= 1 - \frac{1}{\left(\frac{Z}{Y}\right)^2} = 1 - \frac{1}{(1+.01)^2}$$

$$= 1-(1+.01)^{-2} = 1-(1-.02+.0003-\cdots)$$

$$= 0.0197 \qquad (2-4\zeta^2) = 0.310$$

$$0.0197\, r^4 - 0.310\, r^2 + 1 = 0$$

$$\frac{\omega}{\omega_n} = r = \begin{cases} 2.125 \\ 3.350 \end{cases}$$

∴ Lowest freq. for 1% accuracy = 3.35 × 4.75 = $\underline{\underline{15.9 \text{ cps}}}$.

(b) For 2% above plot indicates we must use $\frac{Z}{Y} = 0.98$

$$-0.0412\, r^4 - 0.310\, r^2 + 1 = 0 \quad , \quad r = 1.57$$

$$1.57 \times 4.75 = \underline{\underline{7.45 \text{ cps}}}$$

3-43 $$Z = \frac{\left(\frac{\omega}{\omega_n}\right)^2 Y}{1-\left(\frac{\omega}{\omega_n}\right)^2} \qquad Y = Z\left[\left(\frac{\omega_n}{\omega}\right)^2 - 1\right]$$

$$= 0.052\left[\left(\frac{1}{4}\right)^2 - 1\right] = -0.0488 \text{ cm}$$

3-44 sensitivity = 0.096 rms volts/cm per sec.

0.024 rms volts measured ∴ cm per sec vel = $\frac{.024}{.096} = 0.250$

$\omega = 2\pi \times 30 = 188.49$ r/s.

(a) vel = 0.250 = ωx = 188.49 x_{cm}. $x = \underline{\underline{0.001326}}$ rms cm.

(b) ampl = 0.40 cm peak

cannot be used since 0.40 cm exceeds clearance of 0.30 cm

3-45 sensitivity = 40 mv/cm per s.

$1\ g$ accel $= 9.81\ m/s^2 = 981\ cm/s^2$ accel $= \omega^2 Y$

(a) at 10 cps $\quad \omega = 2\pi 10 = 62.83$ r/s   vel $= \omega Y$.

$$\omega Y = \frac{981}{62.83} = 15.61 \text{ cm/s}.$$

$$\text{Output volts} = 40 \times 15.61 = \underline{\underline{624.5 \text{ mv}}}$$

(b) at 2000 cps $\quad \omega = 12566$

$$\omega Y = \frac{981}{12566} = 0.07806 \text{ cm/s}.$$

$$\text{Output volts} = 40 \times .07806 = \underline{\underline{3.123 \text{ mv}}}$$

3-46 For vel. pick-up $\frac{\omega}{\omega_n} \gg 1 \quad \therefore \quad Z = Y$

Volts generated by instrument $= \omega Z = \omega Y$

3-47 Sensitivity = 30 mv/cm. per s.

3 mv = accuracy Limit $= 30\,(\omega Z) \quad \therefore \quad \omega Z = 0.10$ cm/s = Limiting vel.

$$v_{min} = 0.10 = \frac{981}{2\pi f} \quad \therefore \quad f = \underline{\underline{1561 \text{ cps}}} = \text{upper freq. Limit}.$$

$$\text{At } f = 200 \quad v = \frac{981}{2\pi \times 200} = 0.7807$$

$$\text{instr. reading} = 30 \times .7807 = \underline{\underline{23.42 \text{ mv}}}.$$

3-48 $\quad C = 450$ pF $\qquad Q = 18$ pC/g

$$C_{cable} = 5 \times 50 = 250 \text{ pF}$$

$$E = \frac{Q}{C} = \frac{18}{450} = 0.040 \text{ volts/g} = \underline{\underline{40 \text{ mv/g}}} \text{ open circuit}$$

With cable

$$E = 40 \times \frac{450}{450 + 250} = \underline{\underline{25.7}} \text{ mv/g}$$

3-49

$$\frac{W_d}{U} = \frac{\pi c \omega X^2}{\frac{1}{2} k X^2} = \frac{2\pi c \omega}{k}$$

$$c = \zeta c_c = \zeta 2\sqrt{km}$$

$$\frac{W_d}{U} = 2\pi \cdot 2\zeta \sqrt{km} \cdot \frac{\omega}{k} = 4\pi\zeta \frac{\omega}{\omega_n}$$

3-50

$$\frac{W_d}{U} = 4\pi\zeta\frac{\omega}{\omega_n} , \qquad \zeta \cong \frac{\delta}{2\pi}$$

$$= 2\delta \frac{\omega}{\omega_n}$$

3-51

$$\eta = \frac{W_d}{2\pi U} = \frac{1}{2\pi}\left(4\pi\zeta\frac{\omega}{\omega_n}\right) = 2\zeta\frac{\omega}{\omega_n}$$

but $2\zeta = \gamma$ at resonance

$$\therefore \eta = \gamma\frac{\omega}{\omega_n} = \gamma \quad \text{at resonance}$$

3-52

$$x = \frac{(1-r^2)}{(1-r^2)^2 + (2\zeta r)^2} , \qquad y = \frac{-2\zeta r}{(1-r^2)^2 + (2\zeta r)^2}$$

$$y + \frac{1}{2(2\zeta r)} = \frac{-2\zeta r}{(1-r^2)^2 + (2\zeta r)^2} + \frac{1}{2(2\zeta r)} = \frac{-2(2\zeta r)^2 + (1-r^2)^2 + (2\zeta r)^2}{2(2\zeta r)\left[(1-r^2)^2 + (2\zeta r)^2\right]}$$

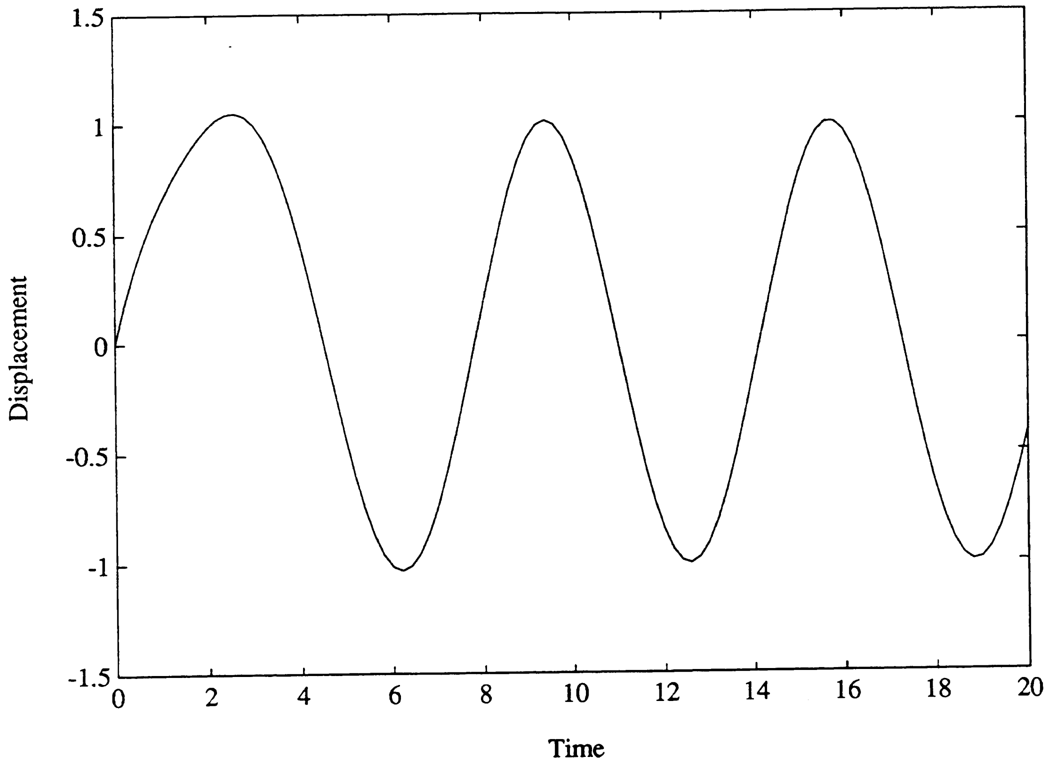
$$= \frac{(1-r^2)^2 - (2\zeta r)^2}{2(2\zeta r)\left[(1-r^2)^2 + (2\zeta r)^2\right]}$$

$$\therefore \quad x^2 + \left(y + \frac{1}{2(2\zeta r)}\right)^2 = \frac{(1-r^2)^2}{\left[(1-r^2)^2 + (2\zeta r)^2\right]} + \left\{\frac{(1-r^2)^2 - (2\zeta r)^2}{2(2\zeta r)\left[(1-r^2)^2 + (2\zeta r)^2\right]}\right\}$$

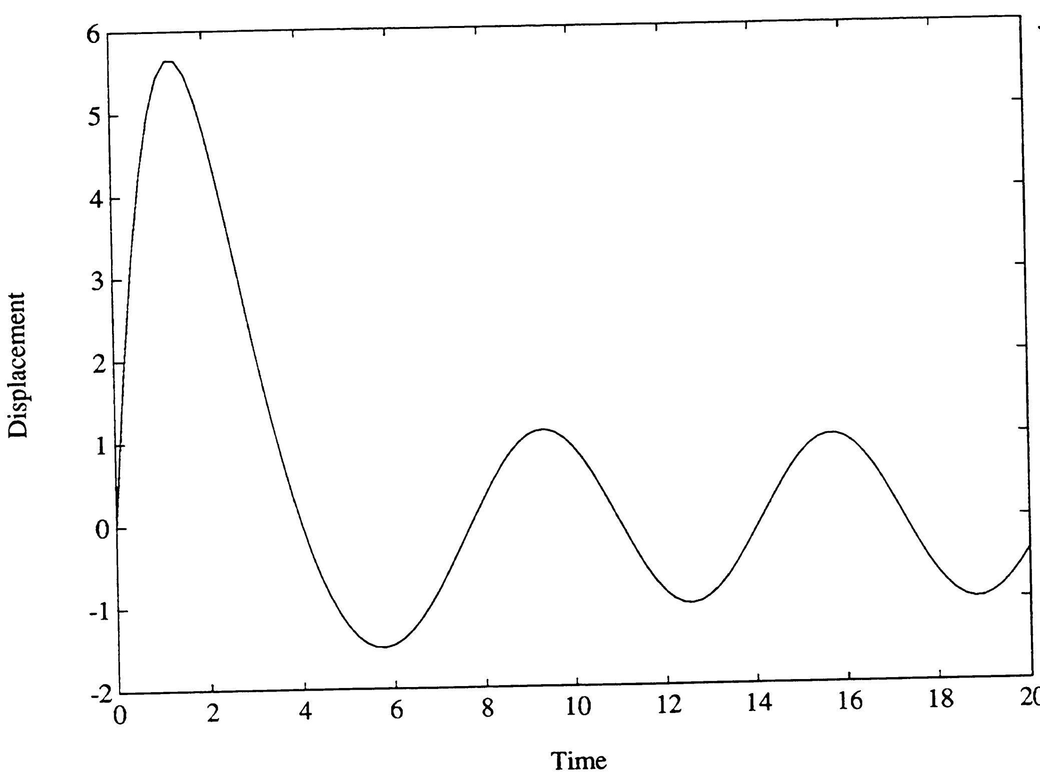
$$= \frac{1}{4(2\zeta r)^2}$$

3-5-3

3-53 ai



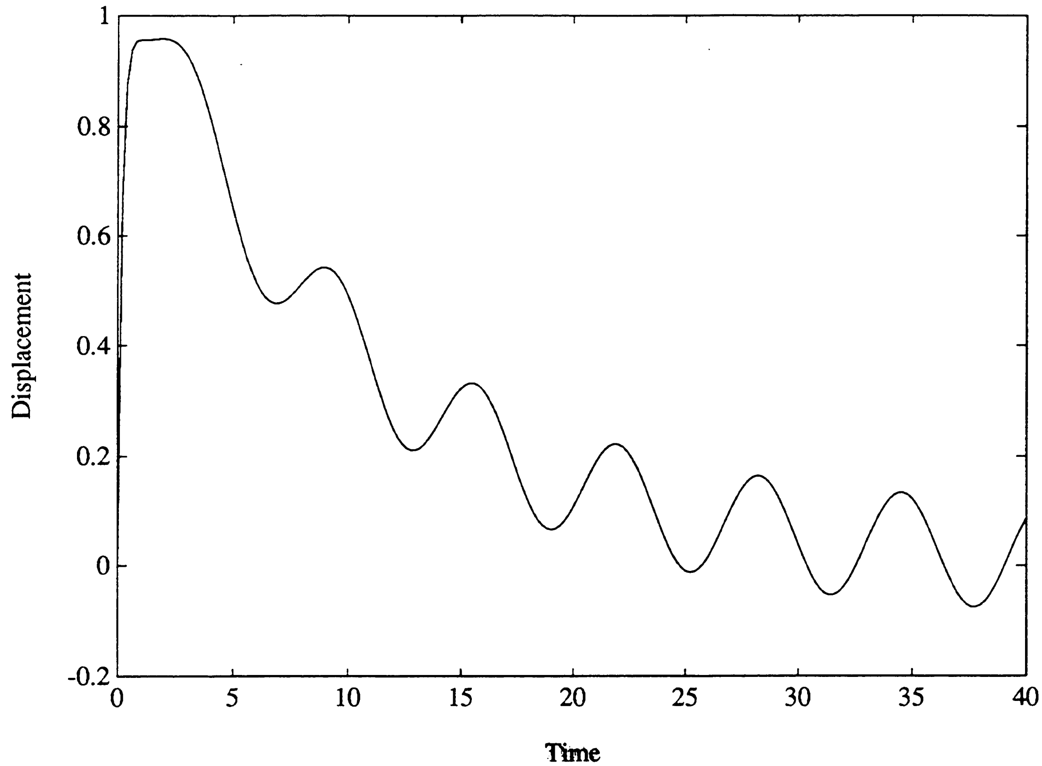
3-53 aii



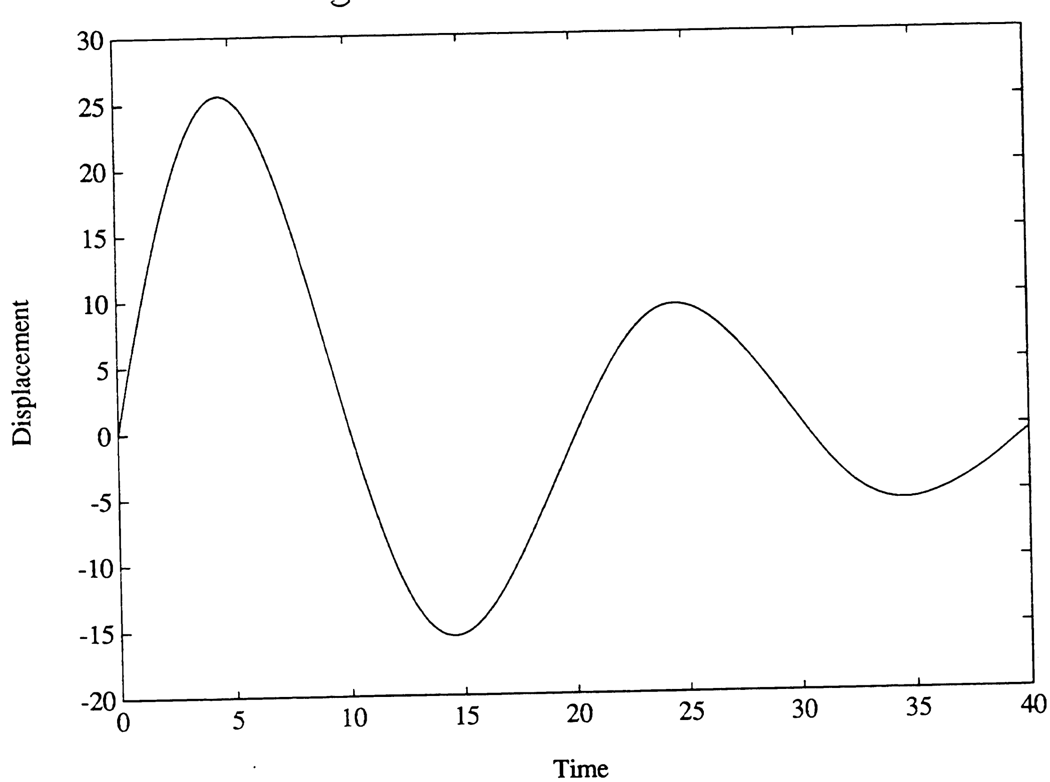
The initial velocity effects the amplitude of the response

3-53 cont

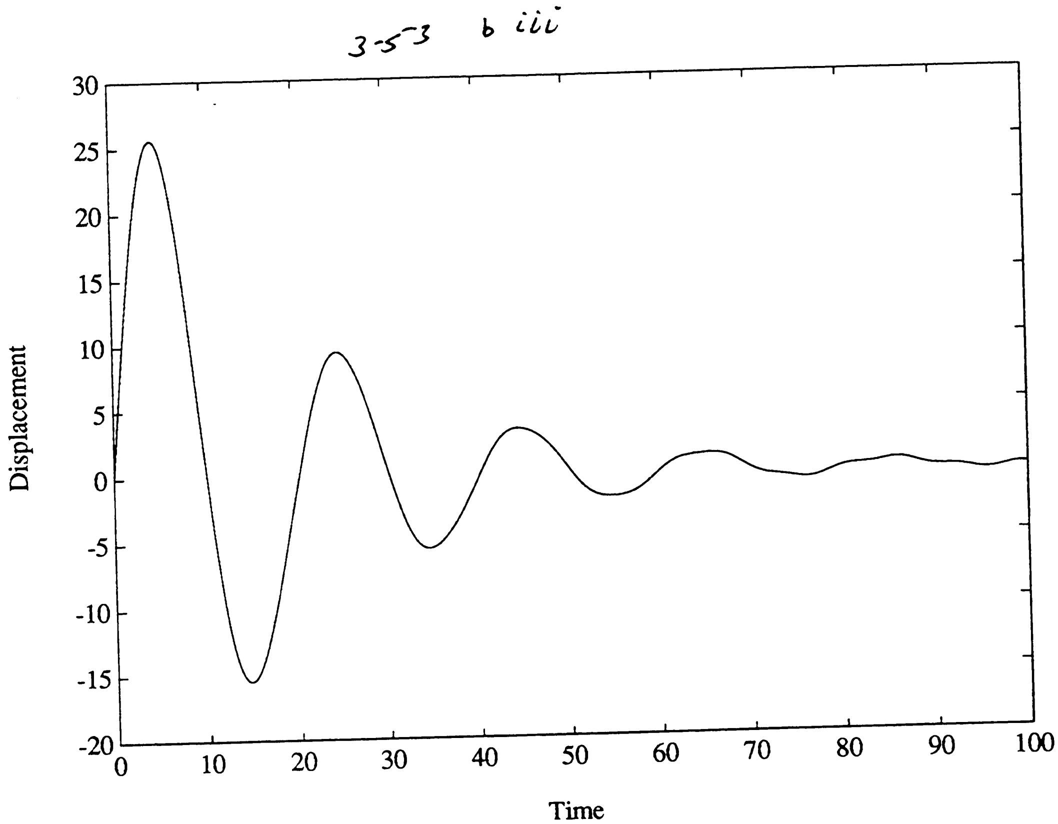
3-53 bc



3-53 b ii



3-5-3 cont



The first system has a larger damping coefficient so it damps out the transient solution faster then the second problem

3-54

$$m\ddot{x} + c\dot{x} + kx = F_1 \sin(\omega_1 t) + F_2 \sin(\omega_2 t)$$

Solution of this equation is written as

$$X(t) = X_c(t) + X_p(t) \quad , \text{ where },$$

$X_c(t)$ is the solution of the homogenous equation, and $X_p(t)$ is the particular solution. Since the differential equation is linear $X_p(t) = X_{p_1}(t) + X_{p_2}(t)$, where,

$$X_{p_i}(t) = X_i \sin(\omega_i t - \phi_i) \qquad \text{with}$$

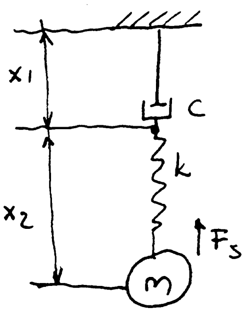
$$\frac{X_i \cdot k}{F_i} = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega_i}{\omega_n}\right)^2\right]^2 + \left[2\zeta\left(\frac{\omega_i}{\omega_n}\right)\right]^2}} \quad \text{and} \quad \tan\phi_i = \frac{2\zeta\left(\frac{\omega_i}{\omega_n}\right)}{1 - \left(\frac{\omega_i}{\omega_n}\right)^2}$$

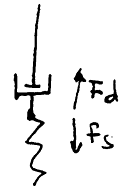
for $i = 1, 2$.

The plots of $\frac{X_i k}{F_i}$ versus $\frac{\omega_i}{\omega_n}$, $i=1,2$ are the same the one $\left(\frac{Xk}{F_0} \text{ versus } \frac{\omega}{\omega_n}\right)$ in the text

$X_c(t)$ can be the critically damped, overdamped or underdamped depending on the value of ζ. The solution to each of these cases is given in the text.

3-55





$$F_d = -F_s$$

$$k(x_2 - x_{20}) = c\dot{x}_1$$

x_{20} – unstretched length of the spring

$$m(\ddot{x}_1 + \ddot{x}_2) = F_s = k(x_2 - x_{20})$$

$$c\ddot{x}_1 = k\dot{x}_2$$

$$\ddot{x}_1 = \frac{k}{c}\dot{x}_2$$

$$m\left(\frac{k}{c}\dot{x}_2 + \ddot{x}_2\right) = k(x_2 - x_{20})$$

Let $z = x_2 - x_{20}$

$$m\ddot{z} = -m\frac{k}{c}\dot{z} - kz$$

$$c_{eff} = m\frac{k}{c}$$

Note: c_{eff} inversely proportional to c!
– with bigger c the dashpot becomes stiffer.

The equation of motion

$$m\ddot{z} + m\frac{k}{c}\dot{z} + kz = F_0 \sin\omega t$$

Energy dissipated by cycle:

$$W_d = \oint c_{eff}\dot{x}\,dx = \oint c_{eff}(\dot{x})^2 dt$$

Integration with $\dot{x} = \omega X \cos(\omega t - \phi)$, where X is the amplitude of motion gives

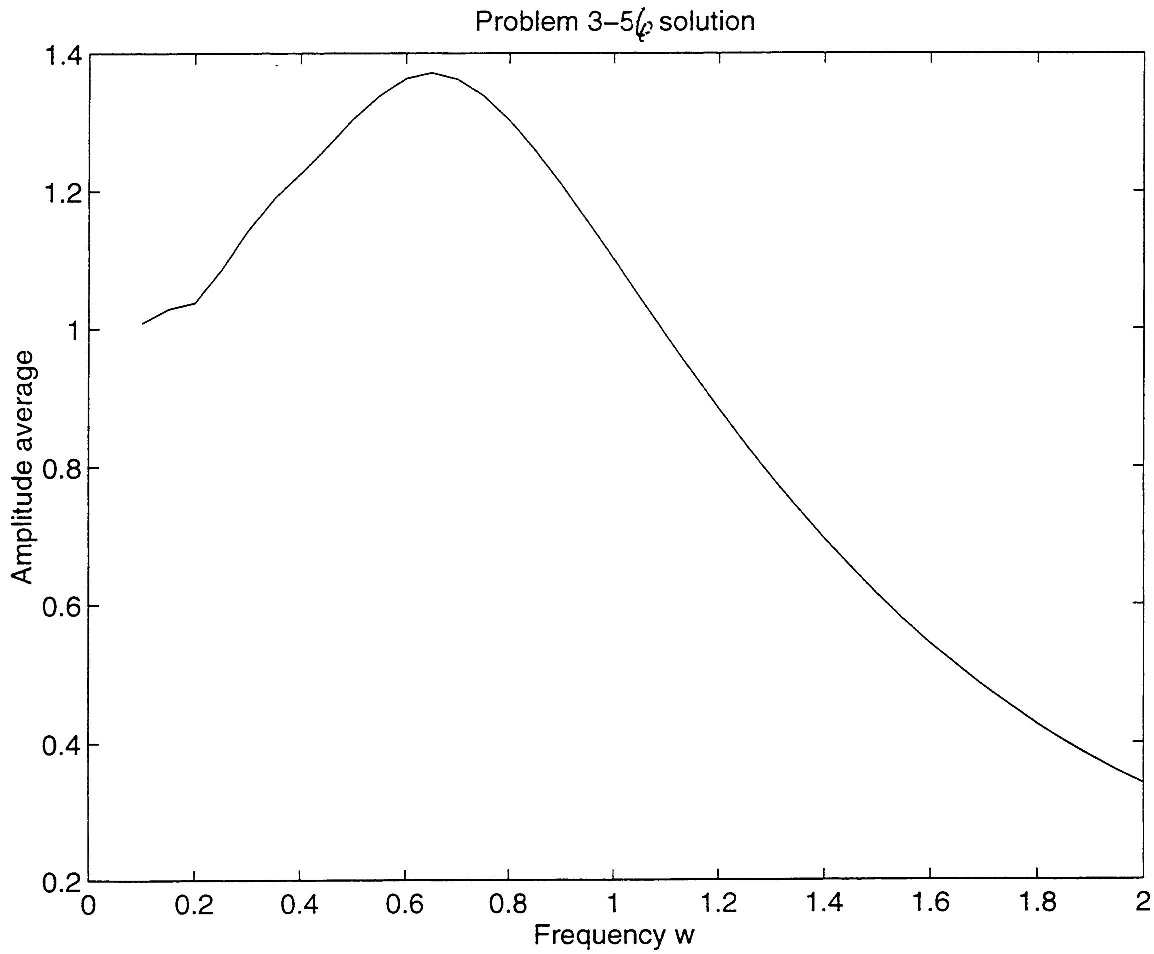
$$W_d = \pi c_{eff}\,\omega X^2 = \pi\frac{mk}{c}\omega X^2$$

3-56

```
clear
global m k a F w
'Problem  3-56 Solution'
x0(1)=0;x0(2)=0;t0=0;
m=input('Enter the value of the mass (kg) : ')
k=input('Enter the value of the spring stiffness : ')
a=input('Enter the value of the damping coefficient a : ')
F=input('Enter the amplitude of the forcing function : ')
nc=input('Enter the desired number of cycles: ')
w1=input('Enter the lower bound of the forcing frequency w : ')
w2=input('Enter the upper bound of the forcing frequency w : ')
iw=input('Enter the desired increment of the forcing frequency w : ')
'Wait ..., this might take some time'
t0=0;l=0;
for  om=w1:iw:w2
tf=nc*2*pi/om;
l=l+1;wf(l)=om;w=om;
clear t x ampl
[t,x]=ode45('p350d',t0,tf,x0);
mx=size(x);n=mx(1,1);
sum=0;j=0;
for  i=1:n-1
    if (x(i+1,2)<0 |  x(i+1,2)==0) & x(i,2)>0
      j=j+1;
           ampl(j)=x(i,1)+(x(i+1,1)-x(i,1))/(x(i+1,2)-x(i,2))*(-x(i,2));
      sum=sum+ampl(j);
   end
end
avamp(l)=sum/j;
end
plot(wf,avamp);xlabel('Frequency  w');ylabel('Amplitude  average')
title('Problem  3-56  solution')
```

```
function  xdot=p356d(t,x)
global m k a F w
if x(2)>0 | x(2)==0
   xdot=[x(2);
           (1/m)*(-a*x(2)^2-k*x(1)+F*sin(w*t))];
else
   xdot=[x(2);
           (1/m)*(a*x(2)^2-k*x(1)+F*sin(w*t))];
end
```

3-56 Cont.



Problem 3–56 solution
Amplitude average
Frequency w
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

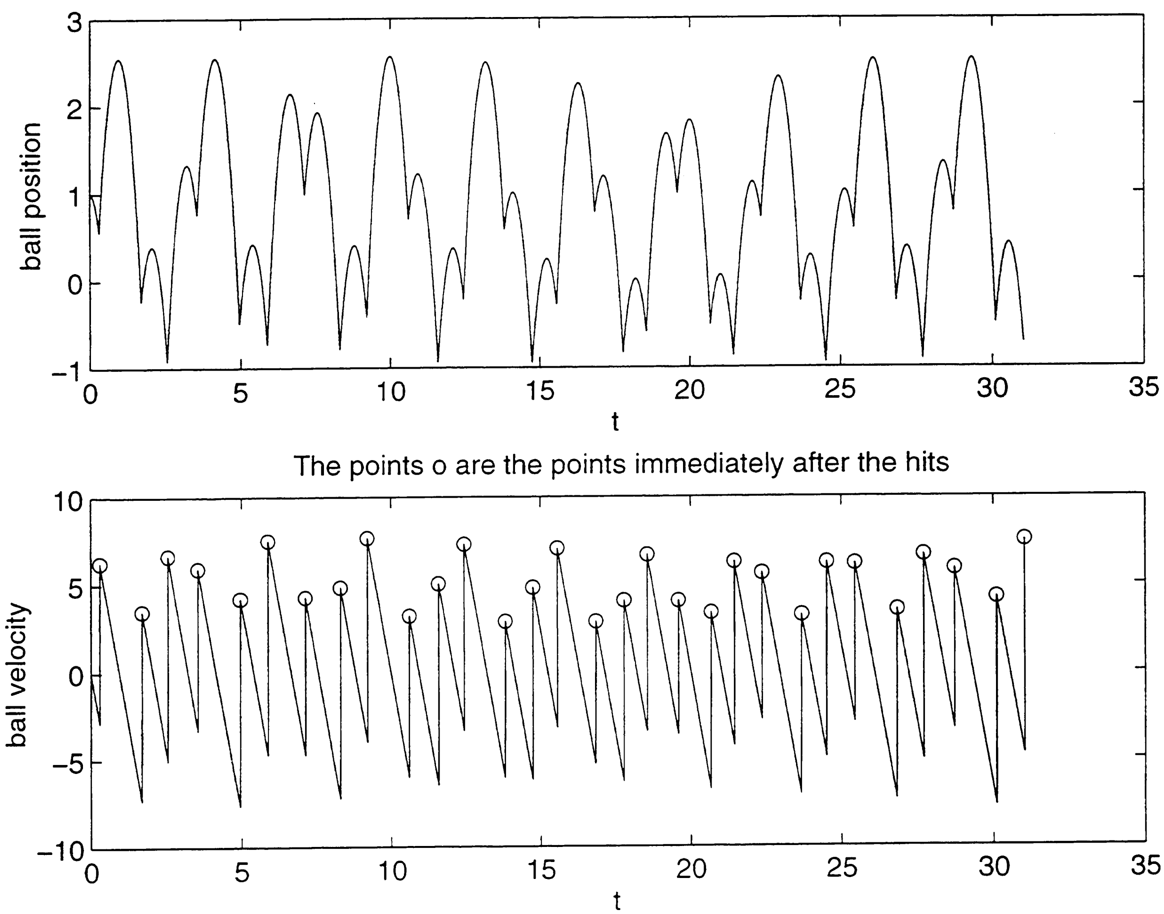
3-57

```
'Problem 3-57 Solution'
clear
x0=input('Enter the value of the initial position of the ball >0 : ')
u0=input('Enter the value of the initial velocity of the ball : ')
N=input('Enter the desired number of hits : ')
w=input('Enter the value of the frequency of the table : ')
'Wait...'
inc=(2*pi)/(300*w);
g=9.81;
t0=0;t(1)=0;x(1)=x0;u(1)=u0;y(1)=0;
j=1;
for i=1:N
    while x(j)>y(j) | x(j)==y(j)
        j=j+1;
          t(j)=t(j-1)+inc;
             x(j)=x0+u0*(t(j)-t0)-0.5*g*(t(j)-t0)^2;
          y(j)=sin(w*t(j));
          u(j)=u0-g*(t(j)-t0);
   end
t(j)=t(j-1)+(y(j-1)-x(j-1))*(t(j)-t(j-1))/((x(j)-y(j))-(x(j-1)-y(j-1)));
thit(i)=t(j);
x(j)=x0+u0*(t(j)-t0)-0.5*g*(t(j)-t0)^2;
y(j)=sin(w*t(j));
v=w*cos(w*t(j));
uneg(i)=u0-g*(t(j)-t0);
uplus(i)=2*v-uneg(i);u(j)=uplus(i);
subplot(2,1,2);plot(thit(i),uplus(i),'o');hold on
u0=uplus(i);t0=t(j);x0=x(j);
end
m=size(x);n=m(1,2);
sum=0;l=0;
for k=2:n-1
if u(k)<0 & u(k-1)>0
     amp=x(k-1)-u(k-1)*(x(k)-x(k-1))/(u(k)-u(k-1));
  sum=sum+amp; l=l+1;
elseif u(k+1)<0 & u(k)==0 & u(k-1)>0;
  sum=sum+x(k);l=l+1;
end
end
'The average amplitude of the ball is'
sum/l
subplot(2,1,1);plot(t,x); xlabel('t');ylabel('ball position')
subplot(2,1,2);plot(t,u); xlabel('t');ylabel('ball velocity')


  title(' The points o are the points immediately after the hits')
```

3-57 cont

Initial position = 1
Initial velocity = 0
Frequency $W = 2$
$N = 30$



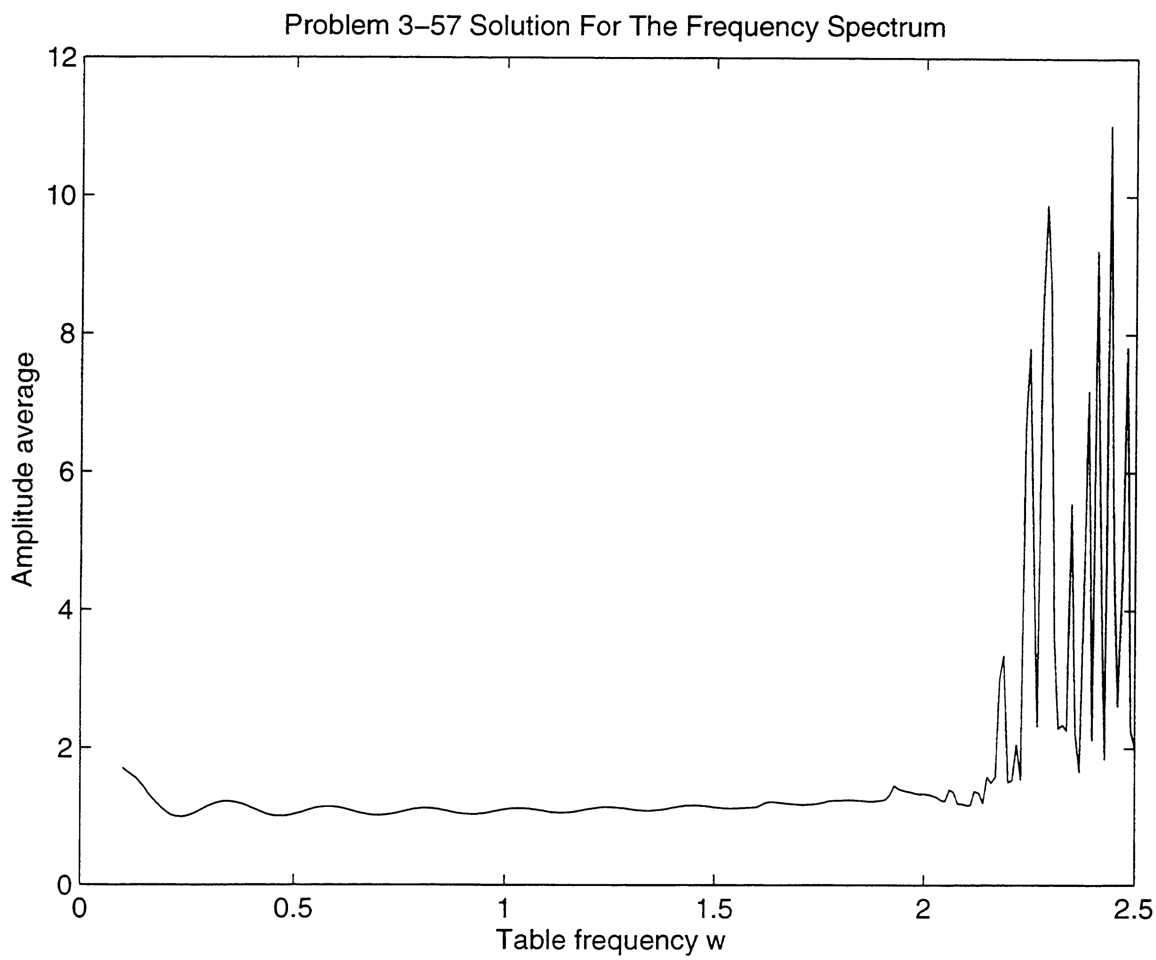
3-57 cont

```
'Problem 3-57 Solution For The Frequency Spectrum'
clear
x0=input('Enter the value of the initial position of the ball >0 : ')
u0=input('Enter the value of the initial velocity of the ball : ')
N=input('Enter the desired number of hits : ')
w1=input('Enter the lower bound of the table frequency w : ')
w2=input('Enter the upper bound of the table frequency w : ')
iw=input('Enter the desired increment of the table frequency w : ')
'Wait..., this might take a long time'
x00=x0;u00=u0;
nw=0;
for  w=w1:iw:w2
nw=nw+1;wf(nw)=w;
clear t x u y
inc=(2*pi)/(300*w);
g=9.81;
x0=x00;u0=u00;
t0=0;t(1)=0;x(1)=x0;u(1)=u0;y(1)=0;
j=1;
for i=1:N
     while x(j)>y(j) | x(j)==y(j)
        j=j+1;
          t(j)=t(j-1)+inc;
              x(j)=x0+u0*(t(j)-t0)-0.5*g*(t(j)-t0)^2;
          y(j)=sin(w*t(j));
           u(j)=u0-g*(t(j)-t0);
    end
t(j)=t(j-1)+(y(j-1)-x(j-1))*(t(j)-t(j-1))/((x(j)-y(j))-(x(j-1)-y(j-1)));
thit(i)=t(j);
x(j)=x0+u0*(t(j)-t0)-0.5*g*(t(j)-t0)^2;
y(j)=sin(w*t(j));
v=w*cos(w*t(j));
uneg(i)=u0-g*(t(j)-t0);
uplus(i)=2*v-uneg(i);u(j)=uplus(i);
u0=uplus(i);t0=t(j);x0=x(j);
end
m=size(x);n=m(1,2);
sum=0;l=0;
for  k=2:n-1
if u(k)<0 & u(k-1)>0
      amp=x(k-1)-u(k-1)*(x(k)-x(k-1))/(u(k)-u(k-1));
   sum=sum+amp; l=l+1;
elseif u(k+1)<0 & u(k)==0 & u(k-1)>0;
```

3-57 cont

```
   sum=sum+x(k);l=l+1;
end
end
avamp(nw)=sum/l;
end
plot(wf,avamp);xlabel('Table frequency w');ylabel('Amplitude average');
title('Problem 3-57 Solution For The Frequency Spectrum')
```

$x_0 = 1$
$u_0 = 0$
$N = 30$
$w_1 = 0.1$
$w_2 = 2.5$
$dw = 0.01$



4-1

Eq. (4.1-6) for impulsive response

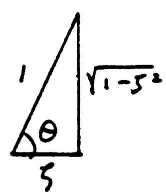
$$x = \frac{\hat{F}}{m\omega_n\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t} \sin\sqrt{1-\zeta^2}\,\omega_n t$$

For max response $\frac{dx}{dt} = 0$

$$\frac{dx}{dt} = \frac{\hat{F}}{m\omega_n\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t}\left\{-\zeta\omega_n \sin\sqrt{1-\zeta^2}\,\omega_n t + \sqrt{1-\zeta^2}\,\omega_n \cos\sqrt{1-\zeta^2}\,\omega_n t\right\} = 0$$

$$\therefore \quad \tan\sqrt{1-\zeta^2}\,\omega_n t = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

4-2



Let $\theta = \sqrt{1-\zeta^2}\,\omega_n t$

$$\tan\theta = \frac{\sqrt{1-\zeta^2}}{\zeta} \quad \text{from Prob 4-1}$$

$$\sin\theta = \sqrt{1-\zeta^2}$$

Eq. (4.1-6) becomes $\quad x_{peak} = \frac{\hat{F}}{m\omega_n\cancel{\sqrt{1-\zeta^2}}}\, e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}} \times \cancel{\sqrt{1-\zeta^2}}$

$$\frac{x_{peak}\sqrt{km}}{\hat{F}} = \exp\left(-\frac{\zeta}{\sqrt{1-\zeta^2}}\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

4-3

From Ex 4.2-1

$$\frac{xk}{F_0} = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos\left(\sqrt{1-\zeta^2}\,\omega_n t - \psi\right)$$

$\tan\psi = \frac{\zeta}{\sqrt{1-\zeta^2}} \qquad$ For peak response $\frac{d}{d(\omega_n t)}\left(\frac{xk}{F_0}\right) = 0$

$$\left\{-\zeta\cos\left(\sqrt{1-\zeta^2}\,\omega_n t - \psi\right) - \sqrt{1-\zeta^2}\,\sin\left(\sqrt{1-\zeta^2}\,\omega_n t - \psi\right) = 0\right.$$

$$\therefore \quad \tan\left(\sqrt{1-\zeta^2}\,\omega_n t - \psi\right) = \frac{-\zeta}{\sqrt{1-\zeta^2}} \qquad \text{expand}$$

$$\frac{\tan\sqrt{1-\zeta^2}\,\omega_n t - \tan\psi}{1 + \tan\sqrt{1-\zeta^2}\,\omega_n t\cdot\tan\psi} = \frac{-\zeta}{\sqrt{1-\zeta^2}}$$

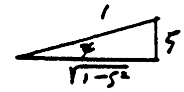
$$\tan\sqrt{1-\zeta^2}\,\omega_n t - \cancel{\frac{\zeta}{\sqrt{1-\zeta^2}}} = -\frac{\zeta}{\sqrt{1-\zeta^2}}\left[\cancel{1} + \frac{\zeta}{\sqrt{1-\zeta^2}}\tan\sqrt{1-\zeta^2}\,\omega_n t\right]$$

$$\tan\sqrt{1-\zeta^2}\,\omega_n t\left[1 + \frac{\zeta^2}{1-\zeta^2}\right] = \tan\sqrt{1-\zeta^2}\,\omega_n t = 0 \quad \therefore \quad \omega_n t = \frac{\pi}{\sqrt{1-\zeta^2}}$$

4-4 Subst. $\omega_n t = \frac{\pi}{\sqrt{1-\zeta^2}}$ into Eq. for peak $\frac{xk}{F_0}$

$$\left(\frac{xk}{F_0}\right)_{peak} = 1 - \frac{1}{\sqrt{1-\zeta^2}} \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right) \cos(\pi - \psi)$$

$$= 1 + \frac{1}{\sqrt{1-\zeta^2}} \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right) \cos\psi$$



$\cos\psi = \sqrt{1-\zeta^2}$

$$\left(\frac{xk}{F_0}\right)_{peak} = 1 + \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)$$

4-5

From Eq. 4.4-4 the response during the first step function is

$$\frac{kx}{F_0} = (1 - \cos\omega_n t)$$

At time t_1 the displacement and velocity are

$$\left(\frac{kx}{F_0}\right)_1 = (1 - \cos\omega_n t_1)$$

$$\left(\frac{k\dot{x}}{F_0}\right)_1 = \omega_n \sin\omega_n t_1$$

which become the initial conditions for the second phase. Since the excitation is zero during the time $t > t_1$, the above equations are substituted into the general solution for the free vibration

$$x(t) = \frac{\dot{x}(t_1)}{\omega_n} \sin\omega_n t + x(t_1) \cos\omega_n t \qquad t > t_1$$

$$x(t) = \frac{F_0}{k}\left\{\sin\omega_n t_1 \sin\omega_n t - (1 - \cos\omega_n t_1)\cos\omega_n t\right\} \qquad t > t_1$$

$$\frac{k\,x(t)}{F_0} = \left\{-\cos\omega_n t + \cos\omega_n t \cos\omega_n t_1 + \sin\omega_n t \sin\omega_n t_1\right\}$$

$$= \left\{-\cos\omega_n t + \cos\omega_n (t - t_1)\right\} \qquad t > t_1$$

4-6

With zero initial conditions Eq. 4.2-1 gives for the particular integral, the equation

$$x(t) = \int_0^t f(\xi)\sin\omega_n(t-\xi)\,d\xi$$

where $f(\xi)$ is the arbitary excitation force.

The solution for the homogeneous equation is found from Eq. 2.2-6

$$x(t) = \frac{\dot{x}(0)}{\omega_n}\sin\omega_n t + x(0)\cos\omega_n t$$

and the complete solution is the sum of these equations.

$$x(t) = \frac{\dot{x}(0)}{\omega_n}\sin\omega_n t + x(0)\cos\omega_n t + \int_0^t f(\xi)\sin\omega_n(t-\xi)\,d\xi$$

4-7

From Eq. 4.2-1, the response to a unit step function is

$$g(t) = \int_0^t h(t-\xi)\,d\xi$$

Differentiate both sides w.r.t. time

$$\dot{g}(t) = h(t)$$

4-8 Eq. 4.2-1 $\quad x = \int_0^t f(\xi)\,h(t-\xi)\,d\xi$

From Prob. 4-7 $\quad \dot{g}(t-\xi) = h(t-\xi)$

$$x(t) = \int_0^t f(\xi)\,\dot{g}(t-\xi)\,d\xi \qquad \text{Integrate by parts}$$

$$x(t) = -f(\xi)\,g(t-\xi)\Big|_0^t + \int_0^t \dot{f}(\xi)\,g(t-\xi)\,d\xi$$

$$= -f(t)\,g(0) + f(0)\,g(t) + \int_0^t \dot{f}(\xi)\,g(t-\xi)\,d\xi \qquad \text{but } g(0) = 0$$

$$\therefore\ x(t) = f(0)\,g(t) + \int_0^t \dot{f}(\xi)\,g(t-\xi)\,d\xi$$

4-9 $$\bar{x}(s) = \frac{(ms+c)\,x(0) + m\,\dot{x}(0)}{ms^2 + cs + k}$$

$$= \frac{(s+2\zeta\omega_n)\,x(0)}{s^2 + 2\zeta\omega_n s + \omega_n^2} + \frac{\dot{x}(0)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

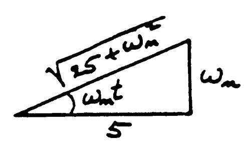
$$\mathcal{L}^{-1}\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{\omega_n\sqrt{1-\zeta^2}}\,e^{-\zeta\omega_n t}\sin\sqrt{1-\zeta^2}\,\omega_n t \qquad \text{see Append. B.}$$

$$\mathcal{L}^{-1}\frac{s}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \text{derivative of above} = e^{-\zeta\omega_n t}\left\{\cos\sqrt{1-\zeta^2}\,\omega_n t - \frac{\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}}\sin\sqrt{1-\zeta^2}\,\omega_n t\right\}$$

$$\therefore\ x(t) = e^{-\zeta\omega_n t}\left\{\frac{\dot{x}(0) + \zeta\omega_n x(0)}{\omega_n\sqrt{1-\zeta^2}}\sin\sqrt{1-\zeta^2}\,\omega_n t + x(0)\cos\sqrt{1-\zeta^2}\,\omega_n t\right\}$$

4-10 $\quad \dot{y}(t) = 20\,u(t) - 100\,t \qquad \therefore\ \ddot{y}(t) = 20\,\delta(t) - 100$

$$\ddot{z} + \omega_n^2 z = -\ddot{y} = 100 - 20\,\delta(t)$$



$$\bar{Z}(s) = \frac{100}{s(s^2+\omega_n^2)} - \frac{20}{(s^2+\omega_n^2)}$$

$$z(t) = \frac{100}{\omega_n^2}(1 - \cos\omega_n t) - \frac{20}{\omega_n}\sin\omega_n t$$

$$\dot{z}(t) = \frac{100}{\omega_n^2}\sin\omega_n t - 20\cos\omega_n t = 0 \qquad \text{for max } z$$

$$\therefore\ \tan\omega_n t = \frac{\omega_n}{5} \qquad \text{(see Fig)}$$

$$z_{max} = \frac{100}{\omega_n}\left[1 - \frac{5}{\sqrt{25+\omega_n^2}}\right] - \frac{20}{\omega_n}\,\frac{\omega_n}{\sqrt{25+\omega_n^2}} \qquad \omega_n = 10$$

4-11 $\qquad F_0 \sin\omega t = F_0 \sin\frac{\pi}{t_1} t \qquad\qquad \omega_n = \frac{2\pi}{\tau}$

$$\ddot{x} + \omega_n^2 x = \frac{F_0}{m} \sin\frac{\pi}{t_1} t$$

Gen Sol. $\quad x(t) = A\sin\omega_n t + B\cos\omega_n t + \dfrac{F_0/m \sin\frac{\pi}{t_1} t}{1-\left(\frac{\pi}{t_1\omega_n}\right)^2}$

For $x(0) = \dot{x}(0) = 0$, $\quad B = 0 \quad$ and $\quad A = \dfrac{-F_0/m\left(\frac{\pi}{t_1\omega_n}\right)}{1-\left(\frac{\pi}{t_1\omega_n}\right)^2}$

With. $\dfrac{\pi}{t_1\omega_n} = \dfrac{\pi}{t_1}\dfrac{\tau}{2\pi} = \dfrac{\tau}{2t_1}$

$$x(t) = \frac{F_0}{m}\left\{\frac{-\frac{\tau}{2t_1}\sin\frac{2\pi t}{\tau}}{1-\left(\frac{\tau}{2t_1}\right)^2} + \frac{\sin\pi\frac{t}{t_1}}{1-\left(\frac{\tau}{2t_1}\right)^2}\right\}$$

$$= \frac{F_0/m}{\left(\tau/2t_1 - 2t_1/\tau\right)}\left\{\sin\frac{2\pi t}{\tau} - \frac{2t_1}{\tau}\sin\frac{\pi t}{t_1}\right\} \qquad t < t_1$$

For $t > t_1$ add same solution with t replaced by $(t-t_1)$

4-12 The triangular force can be represented by

$F_1 = \frac{2F_0}{t_1} t \qquad 0 < t < t_1/2$

$F_2 = -\frac{4F_0}{t_1}(t - t_1/2) + F_1 \qquad t_1/2 < t < t_1$

$F_3 = \frac{2F_0}{t_1}(t - t_1) + F_2 \qquad t_1 < t$

Diff. Eq. for $0 < t < t_1/2$ is

$$\ddot{x} + \omega_n x = \frac{2F_0}{mt_1} t = \left(\frac{2\omega_n^2 F_0}{k t_1}\right) t = ct$$

$\bar{x}_1(s) = \dfrac{c}{s^2(s^2+\omega_n^2)} \qquad \therefore x_1(t) = \dfrac{c}{\omega_n^3}(\omega_n t - \sin\omega_n t) = \dfrac{2F_0}{k}\left(\dfrac{t}{t_1} - \dfrac{\tau}{2\pi t_1}\sin 2\pi\dfrac{t}{\tau}\right)$

For F_2 additional excitation is $-2ct + ct_1 = -2c(t - t_1/2)$

Thus $x_1(t)$ must be suplimented by $-2x_1(t - t_1/2)$

$$x_2(t) = x_1(t) - 2x_1(t - \tfrac{1}{2}t_1) \qquad \tfrac{1}{2}t_1 < t < t_1$$

For F_3 the excitation in addition to F_2 is $c(t-t_1)$

$$x_3(t) = x_2(t) + x_1(t-t_1) \qquad t_1 < t$$

4-11 $\qquad F_0 \sin \omega t = F_0 \sin \frac{\pi}{t_1} t$

$$\ddot{x} + \omega_n^2 x = \frac{F_0}{m} \sin \frac{\pi}{t_1} t \qquad \omega_n = \frac{2\pi}{\tau}$$

Gen Sol. $x(t) = A \sin \omega_n t + B \cos \omega_n t + \dfrac{F_0/m \sin \frac{\pi}{t_1} t}{1 - \left(\frac{\pi}{t_1 \omega_n}\right)^2}$

For $x(0) = \dot{x}(0) = 0$, $B = 0$ and $A = \dfrac{-F_0/m \left(\frac{\pi}{t_1 \omega_n}\right)}{1 - \left(\frac{\pi}{t_1 \omega_n}\right)^2}$

With. $\frac{\pi}{t_1 \omega_n} = \frac{\pi}{t_1} \frac{\tau}{2\pi} = \frac{\tau}{2t_1}$

$$x(t) = \frac{F_0}{m} \left\{ \frac{-\frac{\tau}{2t_1} \sin \frac{2\pi t}{\tau}}{1 - \left(\frac{\tau}{2t_1}\right)^2} + \frac{\sin \frac{\pi t}{t_1}}{1 - \left(\frac{\tau}{2t_1}\right)^2} \right\}$$

$$= \frac{F_0/m}{\left(\tau/2t_1 - 2t_1/\tau\right)} \left\{ \sin \frac{2\pi t}{\tau} - \frac{2t_1}{\tau} \sin \frac{\pi t}{t_1} \right\} \qquad t < t_1$$

For $t > t_1$ add same solution with t replaced by $(t - t_1)$

4-12 $\quad$ The triangular force can be represented by

$F_1 = \frac{2F_0}{t_1} t \qquad 0 < t < t_1/2$

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Diff. Eq. for $0 < t < t_1/2$ is

$$\ddot{x} + \omega_n x = \frac{2F_0}{m t_1} t = \left(\frac{2\omega_n^2 F_0}{k t_1}\right) t = Ct$$

$\bar{x}_1(s) = \dfrac{C}{s^2(s^2 + \omega_n^2)} \quad \therefore x_1(t) = \dfrac{C}{\omega_n^3}(\omega_n t - \sin \omega_n t) = \dfrac{2F_0}{k}\left(\dfrac{t}{t_1} - \dfrac{\tau}{2\pi t_1} \sin 2\pi \dfrac{t}{\tau}\right)$

For F_2 additional excitation is $-2Ct + Ct_1 = -2C(t - t_1/2)$

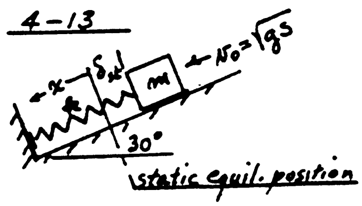
Thus $x_1(t)$ must be supplimented by $-2x_1(t - t_1/2)$

$$x_2(t) = x_1(t) - 2x_1(t - \tfrac{1}{2}t_1) \qquad \tfrac{1}{2}t_1 < t < t_1$$

For F_3 the excitation in addition to F_2 is $C(t - t_1)$

$$x_3(t) = x_2(t) + x_1(t - t_1) \qquad t_1 < t$$

4-13



$$x = x(0)\cos\omega_n t + \frac{\dot{x}(0)}{\omega_n}\sin\omega_n t$$

$$\dot{x} = -\omega_n x(0)\sin\omega_n t + \dot{x}(0)\cos\omega_n t$$

$$k\delta_{st} = \tfrac{1}{2}mg$$

When returned to initial position

$$\left.\begin{aligned} -\delta_{st} &= -\delta_{st}\cos\omega_n t + \frac{v_0}{\omega_n}\sin\omega_n t \\ -v_0 &= \omega_n\delta_{st}\sin\omega_n t + v_0\cos\omega_n t \end{aligned}\right\} \div \text{ by } \cos\omega_n t \text{ and } \div \text{ the two eqs.}$$

$$\frac{\delta_{st}}{v_0} = \frac{-\delta_{st} + \frac{v_0}{\omega_n}\tan\omega_n t}{\omega_n\delta_{st}\tan\omega_n t + v_0}, \qquad \tan\omega_n t = \frac{2\delta_{st}\omega_n v_0}{v_0^2 - \delta_{st}^2\omega_n^2}$$

$$\omega_n^2 = k/m$$

$$\tan\omega_n t = \frac{\sqrt{\frac{m}{k}gs}}{s - \frac{gm}{4k}} \qquad \delta_{st} = \frac{g}{\omega_n^2} \qquad v_0 = \sqrt{gs}$$

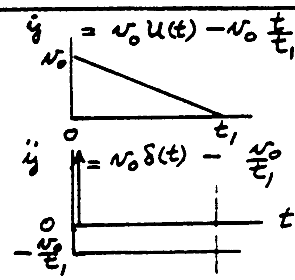
4-14

$$\Delta = \frac{38.6}{6.40} = 6.04'' , \qquad \text{Max displ.} = 2\Delta = 12.08''$$

$$\omega_n = \sqrt{\frac{6.40\times 386}{38.6}} = 8.0 \text{ rad/s.} = \frac{2\pi}{\tau}$$

$$\tau = \frac{2\pi}{8} = 0.784\text{ s.} \qquad t_{\text{max displ}} = \frac{\tau}{2} = 0.392\text{ s.}$$

4-15



$$m\ddot{x} = -k(x-y) - f$$

Let $z = x - y$

$$\ddot{z} + \omega_n^2 z = -\ddot{y} - \frac{f}{m}$$

Gen Sol. from Eq. (4.2-5) is

$$z = \frac{-1}{\omega_n}\int_0^t\left[\ddot{y}(\xi) + \frac{f}{m}\right]\sin\omega_n(t-\xi)\,d\xi$$

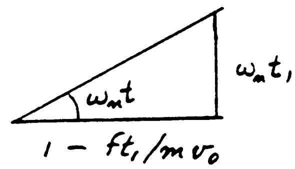
$$= -\frac{v_0}{\omega_n}\int_0^t\left[\delta(t) - \frac{1}{t_1} + \frac{f}{v_0 m}\right]\sin\omega_n(t-\xi)\,d\xi$$

$$= \frac{v_0}{\omega_n}\left\{\frac{1}{\omega_n t_1}\left(1 - \frac{f t_1}{m v_0}\right)(1-\cos\omega_n t) - \sin\omega_n t\right\}$$

4-16 Differentiate z in Prob. 4-15

$$\frac{dz}{dt} = \frac{v_0}{\omega_m}\left\{ \frac{1}{t_1}\left(1 - \frac{ft_1}{mv_0}\right)\sin\omega_m t - \omega_m \cos\omega_m t \right\} = 0$$

$$\therefore \tan\omega_m t = \frac{\omega_m t_1}{1 - \frac{ft_1}{mv_0}}$$



$$\sin\omega_m t = \frac{\omega_m t_1}{\sqrt{(\omega_m t_1)^2 + [1 - ft_1/mv_0]^2}}$$

$$\cos\omega_m t = \frac{(1 - ft_1/mv_0)}{\sqrt{\text{same}}}$$

subst into eq. for Z

$$\frac{\omega_m Z_{max}}{v_0} = \frac{1}{\omega_m t_1}\left(1 - \frac{ft_1}{mv_0}\right)\left\{1 - \frac{\left(1 - \frac{ft_1}{mv_0}\right)}{\sqrt{(\omega_m t_1)^2 + [1 - ft_1/mv_0]^2}}\right\} - \frac{\omega_m t_1}{\sqrt{(\omega_m t_1)^2 + [1 - ft_1/mv_0]^2}}$$

$$= \frac{1}{\omega_m t_1}\left(1 - \frac{ft_1}{mv_0}\right)\left\{1 - \frac{\frac{1}{\omega_m t_1}\left(1 - \frac{ft_1}{mv_0}\right)}{\sqrt{1 + \left[\frac{1}{\omega_m t_1}(1 - ft_1/mv_0)\right]^2}}\right\} - \frac{1}{\sqrt{1 + \left[\frac{1}{\omega_m t_1}(1 - ft_1/mv_0)\right]^2}}$$

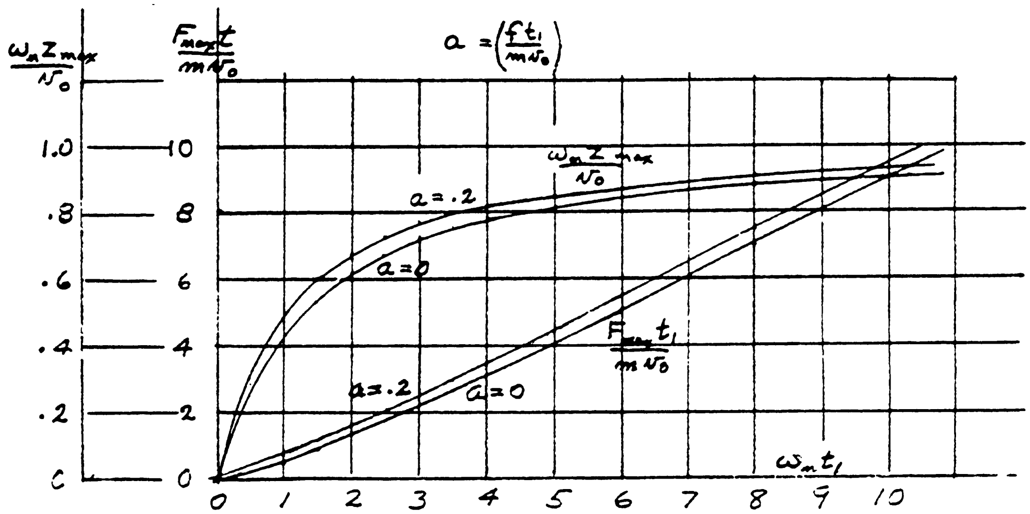
4-17 Let $a = \frac{ft_1}{mv_0}$ and $x = \omega_1 t$ $\quad y = \left|\frac{\omega_m Z_{max}}{v_0}\right|$

$$\left|\frac{\omega_m Z_{max}}{v_0}\right| = \left|\frac{1}{x}(1-a)\left\{1 - \frac{(1-a)}{\sqrt{x^2 + (1-a)^2}}\right\} - \frac{x}{\sqrt{x^2 + (1-a)^2}}\right|$$

$$\left|\frac{F_{max} t_1}{mv_0}\right| = a + \left|(\omega_m t_1)^2\left(\frac{\omega_m Z_{max}}{v_0}\right)\frac{1}{\omega_m t_1}\right| = a + x\left|\left(\frac{\omega_m Z_{max}}{v_0}\right)\right|$$

x	$y_{a=0}$	$\left\|\frac{F_m t_1}{mv_0}\right\|_{a=0}$	$y_{a=.20}$	$\left\|\frac{F_m t_1}{mv_0}\right\|_{a=.2}$
1	.4142	.4142	.4806	.6806
2	.6180	1.2360	.6770	1.554
3	.7208	2.1624	.7683	2.504
4	.7808	3.1232	.8198	3.479
5	.8198	4.0990	.8527	4.466
6	.8471	5.0826	.8755	5.453
8	.8828	7.0624	.9050	7.440
10	.9050	9.050	.9232	9.432

4-17 Cont:



4-18 The response to the ramp function of Fig 4.4-2 is given by Eq. 4.4-3 as

$$x(t) = \frac{F_0}{k}\left[1 - \frac{\sin \omega_n t}{\omega_n t_1} + \frac{1}{\omega_n t_1}\sin \omega_n (t - t_1)\right] \quad t > t_1$$

Differentiating this equation w.r.t. and equating it to zero, the peak time t_p is obtained as

$$\tan \omega_n t_p = \frac{1 - \cos \omega_n t_1}{\sin \omega_n t_1}$$

Since $\omega_n t_p$ must be greater than π, we also obtain

$$\sin \omega_n t_p = -\sqrt{\tfrac{1}{2}(1 - \cos \omega_n t_1)}$$

$$\cos \omega_n t_p = \frac{-\sin \omega_n t_1}{\sqrt{2(1 - \cos \omega_n t_1)}}$$

Substituting these quantities into $x(t)$, the peak amplitude is found as

$$\left(\frac{xk}{F_0}\right)_{max} = 1 + \frac{1}{\omega_n t_1}\sqrt{2(1 - \cos \omega_n t_1)}$$

Letting $\tau = \frac{2\pi}{\omega_n}$ be the period of the oscillator, the above equation is plotted against t_1/τ as in Fig P4-18.

4-19 For small $\frac{t_1}{\tau}$ the sine pulse approaches an impulse

$$\hat{F} = F_0 \int_0^{t_1} \sin \frac{\pi t}{t_1} dt = \frac{2}{\pi} F_0 t_1$$

$$\text{Response} = x = \frac{\hat{F}}{m\omega_n} \sin \omega_n t = \frac{F_0}{k} \frac{4 t_1}{\tau} \sin \frac{2\pi t}{\tau}$$

$$\underline{\underline{\left(\frac{xk}{F_0}\right)_{max} = \frac{4t_1}{\tau}}} \quad \text{at } t_p = \tau/4$$

Peak response must occur at time $\gg t_1 \quad \therefore \frac{t_1}{\tau} \ll \frac{1}{4}$

Let $\alpha = \frac{t_1}{\tau}$, $\xi = \frac{t}{t_1}$ for $t \geq t_1$

$$\left(\frac{xk}{F_0}\right) = \frac{1}{\left(\frac{1}{2\alpha} - 2\alpha\right)} \left[\sin 2\pi\alpha\xi + \sin 2\pi\alpha(\xi - 1)\right]$$

When $\alpha = \frac{1}{2}$ above eq. is indeterminate $= \frac{0}{0}$

$\therefore$ differentiate num. & den. w.r.t. α & divide

$$\left(\frac{xk}{F_0}\right)_{\alpha=\frac{1}{2}} = -\frac{\pi}{2} \cos \pi\xi$$

which is max when $\xi = 1 = \frac{t}{t_1}$

$\therefore t_p = t_1$

$$\text{and} \left(\frac{xk}{F_0}\right)_{\substack{\alpha=\frac{1}{2} \\ \xi=1}} = \frac{\pi}{2} = \underline{\underline{1.57}}$$

4-20 $m = \frac{16.1}{386} = 0.0417$ $\qquad \frac{t_1}{\tau} = \frac{0.40}{0.50} = 0.80$

From Fig P4-21 $\left(\frac{kx}{F_0}\right)_{max} = 1.54$

$\omega_n = \frac{2\pi}{.50} = 4\pi \qquad k = m\omega_n^2 = .0417(4\pi)^2 = 6.585$ #/in

$x_{max} = 1.54 \frac{F_0}{k} \qquad \hat{F} = 2.0$ lb.s. $= \frac{1}{2} \times .40 \times F_0$

$\therefore F_0 = 10$

$$x_{max} = 1.54 \times \frac{10}{6.585} = \underline{\underline{2.339''}}$$

4-21 Differentiate 3rd eq. Prob. 4-12

$$\frac{dx}{dt} = \frac{2F_0}{kt_1}\left\{2\cos\frac{2\pi t_1}{\tau}\left(\frac{t}{t_1}-0.5\right) - \cos\frac{2\pi t_1}{\tau}\left(\frac{t}{t_1}-1.0\right) - \cos\frac{2\pi t}{\tau}\right\} = 0$$

or $$2\cos\frac{2\pi t_1}{\tau}\left(\frac{t_p}{t_1}-0.50\right) - \cos\frac{2\pi t_1}{\tau}\left(\frac{t_p}{t_1}-1\right) - \cos\frac{2\pi t_p}{\tau} = 0$$

which gives $t_p = t_1$

4-22 With suggested subst for $A\cos\phi$ and $A\sin\phi$

$$x = \frac{A}{k}\left\{\sin\omega_n t\cos\phi - \cos\omega_n t\sin\phi\right\} = \frac{A}{k}\sin(\omega_n t - \phi)$$

Max response occurs when $(\omega_n t - \phi) = \frac{\pi}{2}$

$$x_{max} = \frac{A}{k} \quad \text{where} \quad A = \omega_n\sqrt{\left[\int_0^t f(\xi)\sin\omega_n\xi\,d\xi\right]^2 + \left[\int_0^t f(\xi)\cos\omega_n\xi\,d\xi\right]^2}$$

4-23 Let $\frac{\pi}{t_1} = p$, $\quad r = \frac{p}{\omega_n} = \frac{\pi}{t_1}\frac{\tau}{2\pi} = \frac{\tau}{2t_1}$

and rewrite Eq. 4.4-(c)4 for $t < t_1$

$$\left(\frac{kx}{F_0}\right) = \frac{1}{1-r^2}\left[\sin pt - r\sin\omega_n t\right]$$

Differentiate and set equal to zero for max.

$$\left(\frac{k\dot{x}}{F_0}\right) = p\cos pt_m - r\omega_n\cos\omega_n t_m = p\left[\cos pt_m - \cos\omega_n t_m\right] = 0$$

Use identity

$$\cos pt_m - \cos\omega_n t_m = -2\sin\tfrac{1}{2}(p+\omega_n)t_m \cdot \sin\tfrac{1}{2}(p-\omega_n)t_m = 0$$

which is satisfied when $\frac{1}{2}(p+\omega_n)t_m = n\pi$, $\quad n = 1, 2 \cdots$

Therefore the time t_m corresponding to $\left(\frac{kx}{F_0}\right)_{max}$ is

$$t_m = \frac{2n\pi}{p+\omega_n}$$

Substituting into $\frac{kx}{F_0}$, we have

$$\left(\frac{kx}{F_0}\right)_{max} = \frac{1}{1-r^2}\left[\sin\left(\frac{2n\pi p}{p+\omega_n}\right) - r\sin\left(\frac{2n\pi\omega_n}{p+\omega_n}\right)\right]$$

4-23 part (b) for $t > t_1$

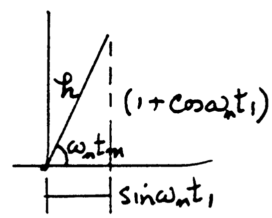
Start with the first equation 4.4-(c) 6

$$\left(\frac{kx}{F_0}\right) = \frac{-r}{1-r^2}\left\{(1+\cos\omega_n t_1)\sin\omega_n t - \sin\omega_n t_1 \cos\omega_n t\right\}$$

where t is measured from t_1. Differentiate and equate to zero to find t_m corresponding to $x(t)_{max}$

$$\left(\frac{k\dot{x}}{F_0}\right) = \frac{-r\,\omega_n}{1-r^2}\left\{(1+\cos\omega_n t_1)\cos\omega_n t_m - \sin\omega_n t_1 \sin\omega_n t_m\right\} = 0$$

$$\therefore \tan\omega_n t_m = \frac{(1+\cos\omega_n t_1)}{\sin\omega_n t_1}$$



$$h = \sqrt{\sin^2\omega_n t_1 + (1+\cos\omega_n t_1)^2} = \sqrt{2(1+\cos\omega_n t_1)}$$

$$= \sqrt{4\cos^2\frac{\omega_n t_1}{2}} = 2\cos\frac{\omega_n t_1}{2}$$

$$\therefore \sin\omega_n t_m = \frac{1+\cos\omega_n t_1}{2\cos\frac{\omega_n t_1}{2}}, \qquad \cos\omega_n t_m = \frac{\sin\omega_n t_1}{2\cos\frac{\omega_n t_1}{2}}$$

Substitute into first eq. for $\left(\frac{kx}{F_0}\right)_{max}$

$$\left(\frac{kx}{F_0}\right)_{max} = \frac{-r}{1-r^2}\left\{\frac{(1+\cos\omega_n t_1)^2 + \sin^2\omega_n t_1}{2\cos\frac{\omega_n t_1}{2}}\right\}$$

$$= \frac{-r}{1-r^2}\left\{\frac{2(1+\cos\omega_n t_1)}{2\cos\frac{\omega_n t_1}{2}}\right\} = \frac{-r}{1-r^2}\left\{\frac{2\cos^2\frac{\omega_n t_1}{2}}{\cos\frac{\omega_n t_1}{2}}\right\}$$

$$= \frac{-r}{1-r^2}\left\{2\cos\frac{\omega_n t_1}{2}\right\} = \frac{2}{r-\frac{1}{r}}\cos\frac{\omega_n t_1}{2}$$

$$= \left(\frac{2}{\frac{\tau}{2t_1} - \frac{2t_1}{\tau}}\right)\cos\frac{\pi t_1}{\tau}$$

4-23 Cont.

$$\left(\frac{kx}{F_0}\right)_{max} = \frac{1}{1-r^2}\left[\sin\left(\frac{2n\pi r}{1+r}\right) - r\sin\left(\frac{2\pi n}{1+r}\right)\right]$$

The first term can be reduced as follows

$$\sin\left(\frac{2n\pi r}{1+r}\right) = \sin\frac{2n\pi(r+1-1)}{1+r} = \sin\left(2n\pi - \frac{2n\pi}{1+r}\right)$$

$$= \sin 2n\pi \cos\frac{2n\pi}{1+r} - \cos 2n\pi \sin\frac{2n\pi}{1+r}$$

$$= -\sin\left(\frac{2n\pi}{1+r}\right)$$

$\therefore$ Replace 2^{nd} term by $-\sin\left(\frac{2n\pi r}{1+r}\right)$

$$\left(\frac{kx}{F_0}\right)_{max} = \frac{(1+r)}{1-r^2}\left[\sin\frac{2n\pi r}{1+r}\right]$$

$$= \frac{1}{1-r}\sin\frac{2n\pi r}{1+r} \qquad \text{where } r = \frac{\tau}{2t_1}$$

$$\left(\frac{kx}{F_0}\right)_{max} = \frac{1}{1-\frac{\tau}{2t_1}}\sin\left[\frac{2n\pi\left(\frac{\tau}{2t_1}\right)}{1+\frac{\tau}{2t_1}}\right]$$

The above equation is plotted as Fig 4.5-5 againt the horizontal scale t_1/τ. Computed values appear below

t_1/τ	$\left(\frac{kx}{F_0}\right)_{max}$ $n=1$
.10	.2165
.25	.866
.30	1.063
.40	1.368
.60	1.694
1.0	1.732
1.5	1.500

t_1/τ	$\left(\frac{kx}{F_0}\right)_{max}$ $n=2$
2.0	1.268
2.5	1.083
3.0	1.170

4-24 Given vel. $= v = v_0\left(U(t) - t/t_1\right)$

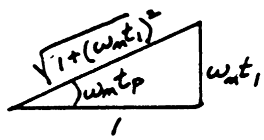
Accel $= a = v_0\left[\delta(t) - 1/t_1\right] = \ddot{y}(t)$

Subst into Eq. 4.2-5

$$z = -\frac{v_0}{\omega_n}\int_0^t \left[\delta(\xi) - \frac{1}{t_1}\right]\sin\omega_n(t-\xi)\,d\xi \qquad 0<t<t_1$$

$$= \frac{v_0}{\omega_n}\left\{-\sin\omega_n t + \frac{1}{\omega_n t_1}\left(1-\cos\omega_n t\right)\right\} \qquad 0<t<t_1$$

$$\frac{dz}{dt} = \frac{v_0}{\omega_n}\left\{-\omega_n\cos\omega_n t + \frac{1}{t_1}\sin\omega_n t\right\} = 0$$



or $\tan\omega_n t_p = \omega_n t_1$ for z_{max}

$$\sin\omega_n t_p = \frac{\omega_n t_1}{\sqrt{1+(\omega_n t_1)^2}}$$

$$\cos\omega_n t_p = \frac{1}{\sqrt{1+(\omega_n t_1)^2}}$$

$$\therefore \frac{z_{max}\,\omega_n}{v_0} = \left\{\frac{1}{\omega_n t_1} - \frac{1}{\omega_n t_1\sqrt{1+(\omega_n t_1)^2}} - \frac{\omega_n t_1}{\sqrt{1+(\omega_n t_1)^2}}\right\}$$

4-25 If $t > t_1$ the solution is

$$z = -\frac{v_0}{\omega_n}\int_0^{t_1}\left[\delta(\xi) - \frac{1}{t_1}\right]\sin\omega_n(t-\xi)\,d\xi$$

$$= \frac{v_0}{\omega_n}\left\{-\sin\omega_n t + \frac{1}{\omega_n t_1}\left[\cos\omega_n(t-t_1) - \cos\omega_n t\right]\right\}$$

4-26 $\ddot{z} + \omega_n^2 z = -\ddot{y} = 100 - 20\delta(t)$

$\tau = \frac{2\pi}{\omega_n} = 0.628$ s.

choose $\Delta t = 0.05$ s

For the impulse response, it is advisable to treat it separately.

Its response is $-\frac{20}{\omega_n}\int_0^t \delta(\xi)\sin\omega_n(t-\xi)d\xi = -2\sin 10t$

Remaining solution can be computed numerically from D.E

(a) $\ddot{z}_i = 100 - 100 z_i$

(b) $\ddot{z}_i \Delta t^2 = z_{i-1} - 2z_i + z_{i+1}$

$\Big\}$ Initial cond. $z_1 = \dot{z}_1 = 0$

Start with $\ddot{z}_1 = 100$ from (a)

From Eq(4.7-8) $z_2 = \frac{1}{2}(0.05)^2 100 = 0.1250$

from (a) $\ddot{z}_2 = 100 - 100(0.1250)$

Find z_3 from Eq. (b) etc.

Alternately, one could use the built in ODE solver in Matlab.

```
%This is the code for problem 4-26
t0=0;
tf=.05;
x0=[0 0 ]';
[t,x]=ode23('prob426',t0,tf,x0);
plot(t,x)
```

```
function xdot=prob426(t,x)
xdot=[100-100*x(2);x(1)];
```

4-27

```
(' This is example 4.7-2 on page 113 in the text')
clear;
clg;
J=0;
T=0;
X=0;
Xdd=0;
F=0;
dt=0.05;
T(1)=0;
F(1)=0;
X(1)=0;
J(1)=1;
for I=2:25,
    J(I)=I;
    T(I)=T(1)+dt*(I-1);
    if I > 2
       if I < 5
          F(I)=500*dt*(I-1);
        else
          if I < 9
             F(I)=200-500*dt*(I-1);
          else
             F(I)=0;
          end
       end
     X(I)=Xdd(I-1)*dt^2-X(I-2)+2*X(I-1);
     Xdd(I)=2*F(I)-16*pi^2*X(I);
     else
        F(I)=500*dt*(I-1);
        X(I)=dt^2*F(I)/(3+8*pi^2*dt^2);
        Xdd(I)=2*F(I)-16*pi^2*X(I);
    end
end
A=['     J      ' '    TIME      ' ' DISPL     ' 'ACCLRTN      ' 'FORCE']
B=[J' T' X' Xdd' F']
plot(T,X)
hold on
plot(T,X,'x')
xlabel('Time, seconds')
ylabel('Diplacement X, m')
hold off
```

4-28 See Ex. 4.2-2 for $y = v_0 e$

Fig P4-28 is for $\dot{y} = 60\, e^{-0.10t}$ applies for $\omega_n t_1$ or $\omega_n t_0$

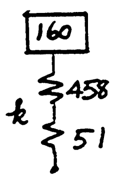
$\therefore\ v_0 = 60\ ,\quad t_0 = 10$

For large $\omega_n t_0$ $\quad \left(\dfrac{2z}{v_0 t_0}\right)_{max} \cong \dfrac{2}{\omega_n t_0}$ a rectangular hyperbola

at $\omega_n t_0 = 100$ Fig 4-28 gives $\left(\dfrac{2z}{v_0 t_0}\right)_{max} = 0.02$

For small $\omega_n t_0$ $\quad \left(\dfrac{2z}{v_0 t_0}\right)_{max} \cong 1.0$

4-29



$$k = \frac{k_1 k_2}{k_1 + k_2} = \frac{458 \times 51}{458 + 51} = 45.89\ \#/in$$

$$\delta_{st} = \frac{160}{45.89} = 3.49''$$

$$\frac{\ddot{x}}{g} = \sqrt{\frac{2h}{\delta_{st}} + 1} = 1.65$$

4-30

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{458 \times 386}{160}} = \sqrt{1105} = 33.24 = \frac{2\pi}{\tau}$$

$$\therefore\ \tau = \frac{2\pi}{33.24} = .1889\ , \quad \frac{t_1}{\tau} = .85$$

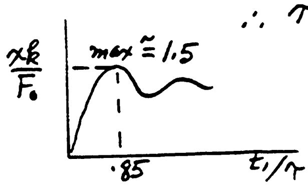
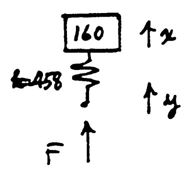
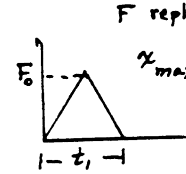


Fig. 4.5-6





F replaced by $m\ddot{y}_{max}$

x_{max} " " z_{max}

4-31 $\quad m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega_n t$

With zero initial conditions, response may be evaluated from (1) (2) or (3) below

(1) $\quad \bar{x}(s) = \dfrac{\omega_n F_0/m}{(s^2+\omega_n^2)(s^2 + 2\zeta\omega_n + \omega_n^2)}$

(2) $\quad x(t) = X_1 e^{-\zeta\omega_n t} \sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \phi_1\right) - \dfrac{F_0 \cos\omega_n t}{c\omega_n}$ $\qquad$ Eq. (3.1-11)

(3) $\quad x(t) = F_0 \displaystyle\int_0^t \sin\omega_n(t-\tau) \cdot \frac{e^{-\zeta\omega_n t}}{m\omega_n\sqrt{1-\zeta^2}} \sin\sqrt{1-\zeta^2}\,\omega_n t$ $\qquad$ Eq. (4.2-2)

Result is:

$$x(t) = \frac{F_0}{c\omega_n}\left\{ \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \sin^{-1}\sqrt{1-\zeta^2}\right) - \cos\omega_n t \right\}$$

For small damping $\delta \cong 2\pi\zeta$

At time $t = \frac{1}{f_n\delta}$, $\zeta\omega_n t = \frac{2\pi\zeta}{\delta} \cong 1.0$ Subst. into Eq. above

$$x = \frac{F_0}{c\omega_n}\left\{ e^{-1}\sin(\omega_n t + 90^\circ) - \cos\omega_n t \right\}$$

$$= \frac{F_0}{c\omega_n}\left\{ e^{-1} - 1 \right\}\cos\omega_n t = (1 - e^{-1})\left(\frac{-F}{c\omega_n}\cos\omega_n t \right)$$

where steady state sol. $= \left(\frac{-F}{c\omega_n}\cos\omega_n t \right)$

66

4-33 Under harmonic force of freq. ω_n, the steady state oscillation is

$$x(t) = \frac{F_0}{c\omega_n}\cos\omega_n t \qquad x(0) = \frac{F_0}{c\omega_n}, \qquad \dot{x}(0) = 0$$

Transient sol. is

$$x(t) = X_1 e^{-\zeta\omega_n t}\sin(\sqrt{1-\zeta^2}\,\omega_n t + \phi_1) \cong X_1 e^{-\zeta\omega_n t}\sin(\omega_n t + \phi_1)$$

for ζ small

$$x(0) = X_1\sin\phi_1 = \frac{F_0}{c\omega_n}$$

$$\dot{x}(0) = X_1[\omega_n\cos\phi_1 - \zeta\omega_n\sin\phi_1] \cong X_1\omega_n\cos\phi_1 = 0$$

$$\therefore \phi_1 = 90^\circ \quad \text{and} \quad X_1 = \frac{F_0}{c\omega_n}$$

Then trans. sol. with above initial cond. is

$$x(t) = \frac{F_0}{c\omega_n} e^{-\zeta\omega_n t}\cos\omega_n t$$

At $\zeta\omega_n t = \frac{2\pi\zeta}{\delta} \cong 1.0$ $\qquad x(t) = e^{-1}\frac{F_0}{c\omega_n}\cos\omega_n t$

see Prob 4-32

4-34

```
%     (' This is example 4.7-1                 in the text')
clear;clg;
J=0;
T=0;
X=0;
dt=0.02;
X(1)=0;
Xd(1)=0;
for I=1:24;
    J(I)=I;
    T(I)=dt*(I-1);
    if I < 6
       F(I)=100;
    elseif I < 11;
       F(I)=-1000*(T(I)-0.10)+100;
    else
       F(I)=0;
    end
    Xdd(I)=0.25*F(I)-500*X(I);
    if I > 1
       X(I+1)=2*X(I)-X(I-1)+dt^2*Xdd(I);
    else
       X(2)=X(1)+dt*Xd(1)+0.5*dt^2*Xdd(1);
    end
end
J(I+1)=I+1;
T(I+1)=dt*I;
['    J     ' '    Time     ' ' Diplacement']
[J' T' X']
plot(T,X,'o')
hold on
t1=0:0.02:0.1;
x1=0.05*(1-cos(22.36*t1));
plot(t1,x1)
hold on
t2=0.1:0.02:0.2;
x2=0.05*(1-cos(22.36*t2))-(0.5*(t2-0.1)-0.02236*sin(22.36*(t2-0.10)));
plot(t2,x2)
hold on
t3=0.2:0.02:0.48;
x3=0.05*(1-cos(22.36*t3))-(0.5*(t3-0.1)-0.02236*sin(22.36*(t3-0.10)))+0.5*(t3-0
2)-0.02236*sin(22.36*(t3-0.20));
plot(t3,x3)
xlabel('t, seconds')
ylabel('Diplacement, m')
 hold off
```

4-35 The Differential Equation for base excitation is

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0$$

let $\quad z = x - y$

$$\ddot{z} + 2\zeta\omega_n \dot{z} + \omega_n^2 z = -\ddot{y}$$

where $\quad -\ddot{y} \quad$ is given by

$$y = y_o \sin\omega t$$

$$-\ddot{y} = \omega^2 y_o \sin\omega t$$

The sine pulse terminates at $\omega t_p = \pi$. The time increment h should be chosen smaller than $t_p/10$ or $\frac{1}{10}\left(\frac{2\pi}{\omega_n}\right)$, whichever is the smallest.

The initial conditions are

$x(0) = X_1 \qquad\qquad y(0) = 0$

$\dot{x}(0) = V_1 \qquad\qquad \dot{y}(0) = y_o\omega$

$$z(0) = X_1$$

$$\dot{z}(0) = V_1 - y_o\omega$$

4-35 cont. The two equations given from the Taylor series are

$$\text{(a)} \qquad z_2 = z_1 + \dot{z}_1 h + \frac{h^2}{2}\left(F_1 - \omega_n^2 z_1 - 2\zeta\omega_n \dot{z}_1\right)$$

$$\text{(b)} \qquad z_1 = z_2 - \dot{z}_2 h + \frac{h^2}{2}\left(F_2 - \omega_n^2 x_2 - 2\zeta\omega_n \dot{z}_2\right)$$

Since the initial conditions z_1, $\dot{z}_1$ and F_1 are given Eq (a) gives z_2. Eq (b) is solved for $\dot{z}_2$ which is

$$\text{(c)} \qquad \dot{z}_2 = \frac{z_2 - z_1 + \frac{h^2}{2}\left(F_2 - \omega_n^2 z_2\right)}{h + h^2\zeta\omega_n}$$

Thus z_2 and $\dot{z}_2$ for the first interval can be calculated from eqs. (a) and (c)

Calculations for $z_3, \dot{z}_3$, z_4 $\dot{z}_4$ etc are now made with (Eq 4.7-2) and Eq (c) generalized to index i as follows

$$\text{(d)} \qquad z_{i+1} = 2z_i - z_{i-1} + h^2\left(F_i - \omega_n^2 z_i - 2\zeta\omega_n \dot{z}_i\right)$$

$$\text{(e)} \qquad \dot{z}_{i+1} = \frac{z_{i+1} - z_i + h^2/2\left(F_{i+1} - \omega_n^2 z_{i+1}\right)}{\left(h + h^2\zeta\omega_n\right)}$$

The program structure is similar to problem 4-34

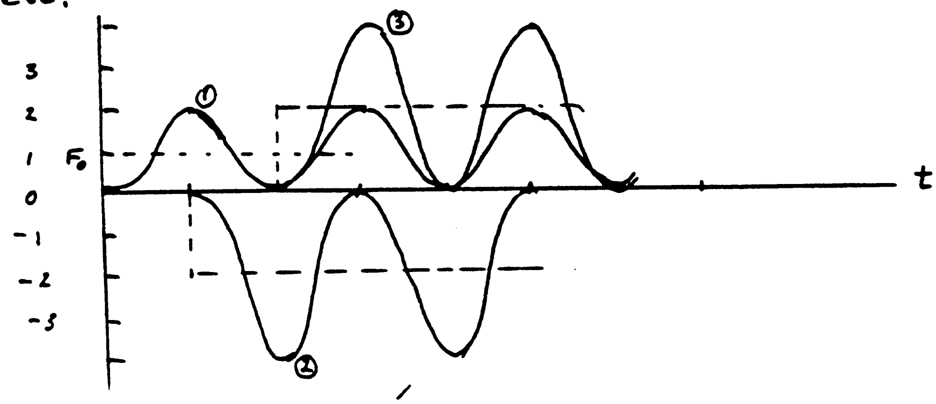
4-36 use superposition of solution to each step function

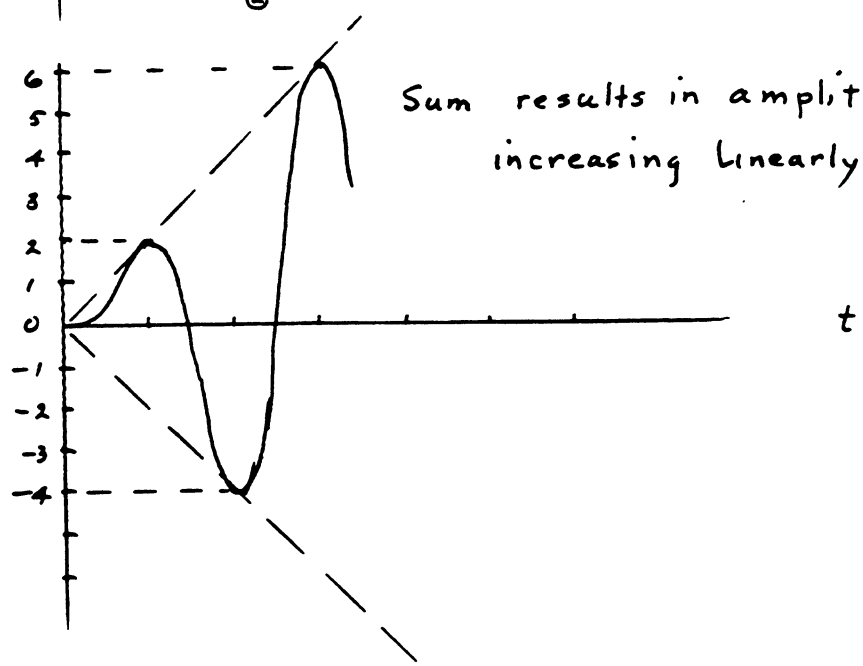
1^{st} step of F_o results in $x(t) = \frac{F_o}{k}(1-\cos\omega_n t)$

2^{nd} step of $-2F_o$ at $t=\frac{\pi}{\omega}$ adds $-\frac{2F_o}{k}\left[1-\cos\omega_n\left(t-\frac{\pi}{\omega}\right)\right]$

3^{rd} step of $+2F_o$ at $t=\frac{2\pi}{\omega}$ adds $+\frac{2F_o}{k}\left[1-\cos\omega_n\left(t-\frac{2\pi}{\omega}\right)\right]$

etc.





Sum results in amplitudes increasing linearly

4-37

$$x_{i+1} = x_i + \dot{x}_i h + \ddot{x}_i \frac{h^2}{2} + \dddot{x}_i \frac{h^3}{6} + \ddddot{x}_i \frac{h^4}{24} + \cdots$$

$$x_{i-1} = x_i - \dot{x}_i h + \ddot{x}_i \frac{h^2}{2} - \dddot{x}_i \frac{h^3}{6} + \ddddot{x}_i \frac{h^4}{24} - \cdots$$

$$x_{i+1} + x_{i-1} = 2x_i + \ddot{x}_i h^2 + \ddddot{x}_i \frac{h^4}{12}$$

$$\ddot{x}_i = \frac{x_{i-1} - 2x_i + x_{i+1}}{h^2} - \ddddot{x}_i \frac{h^2}{12}$$

$$\text{error} = -\frac{\ddddot{x}_i}{12} h^2 = O(h^2)$$

4-38

t	$x = t^3$
.8	.5120
.9	.7290
1.0	1.0000
1.1	1.3310
1.2	1.7280

Exact value of $\dot{x}$ is

$$\dot{x} = 3t$$

$$\therefore \dot{x}(1) = 3.0000$$

Calculation of $\dot{x}$ by finite difference

using $\dot{x}_i = \frac{1}{2h}(x_{i+1} - x_{i-1})$

$h = 0.10$ $\quad \dot{x}_1 = \frac{1}{2 \times .10}(1.3310 - .7290) = 3.010$

$\therefore \text{Error} = .01 = .1^2 = h^2$

$= O(h^2)$

$h = 0.20$ $\quad \dot{x}_1 = \frac{1}{2 \times .20}(1.7280 - .5120) = 3.04$

$\therefore \text{Error} = .04 = .2^2 = h^2$

$= O(h^2)$

4-39

Refer to Prob. 4-38 but use

$$\dot{x}_i = \frac{1}{h}\left(x_i - x_{i-1}\right)$$

$h = .10$ $\quad \dot{x}_1 = \frac{1}{.10}\,(1.0 - .7290) = 2.71$

$\therefore$ Error $= .290 = 2.9\,h = O(h)$

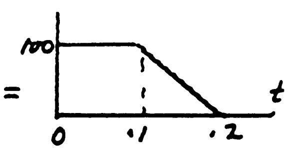
$h = .20$ $\quad \dot{x}_1 = \frac{1}{.20}\,(1.0 - .5120) = 2.44$

$\therefore$ Error $= .560 = 2.8\,h = O(h)$

4-40

D.E. $\quad 4\ddot{x} + 2000\,x = F(t)$

$\therefore\ m = 4, \quad k = 2000$ $\qquad F(t) =$



$$\omega_n = \sqrt{\frac{k}{m}} = 22.36$$

1^{st} sol. due to step function (see Fig 4.7-4)

$$x_1 = \frac{F}{k}(1 - \cos\omega_n t) = \frac{100}{2000}\,(1 - \cos 22.36\,t)$$

$$0 \le t \le 0.10$$

2^{nd} sol. due to ramp function of

$1000\,(t - .10)$ at $t \ge .10$

DE $\quad \ddot{x} + 500\,x = 250\,t'$ $\quad$ where $t' = t - .10$

Lapl. Trans $\quad x(s) = \dfrac{250}{s^2(s^2 + 22.36^2)}$

$$x_2 = \frac{250}{22.36^2}\left[22.36\,t' - \sin 22.36\,t'\right]$$

$$= \tfrac{1}{2}\,t' - .02236 \sin 22.36\,t'$$

4-40 Cont.

$\therefore$ add to first sol x_1, the second sol. which is $x_2 = \frac{1}{2}(t-.10) - .02236 \sin 22.36(t-.10)$

at $t \geq .10$

similarly the third sol. is same as x_2 with $(t-.20)$

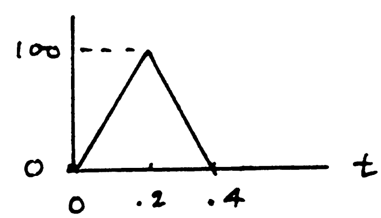
4-41

DE to be solved is

$$\ddot{x} + 16\pi^2 x = 2F(t)$$

Let $y = \dot{x}$, then

$$\dot{y} = f(x,t) = 2F(t) - 16\pi^2 x$$

where $F(t) =$ 

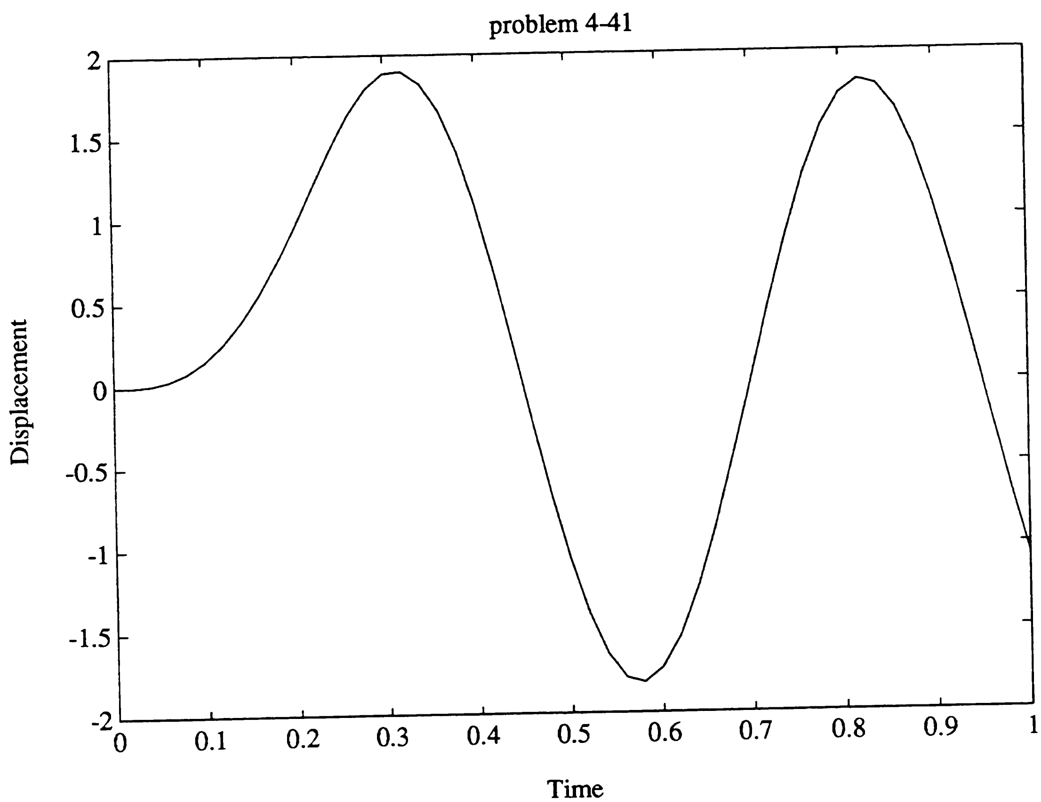
Follow Calc. procedure of Example 4.8-2

$\omega_n = 4\pi = 12.56 = \frac{2\pi}{\tau}$ $\tau = .500$

$\therefore$ suggest $h = .02$

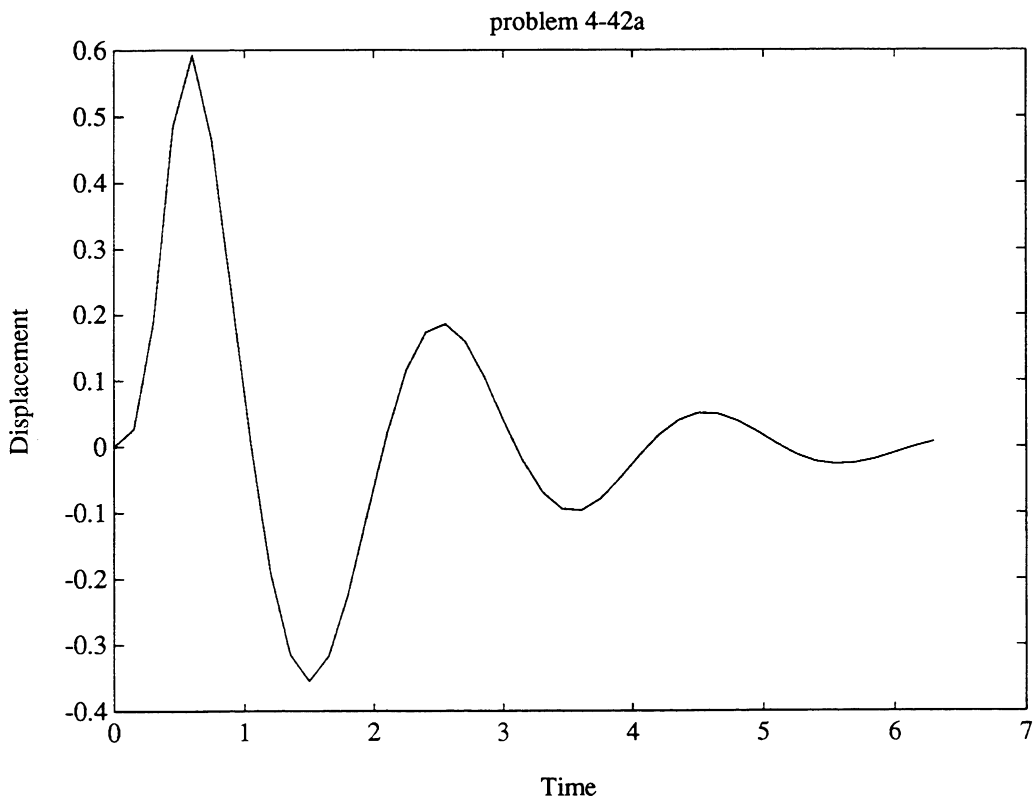
4-41 cont

```
%This is the file f.m for problem 4-41
function [force] = f(t).
 if t<.2
force=500*t;
elseif t<.4
force=200-500*t;
else
force=0;
end
end
```



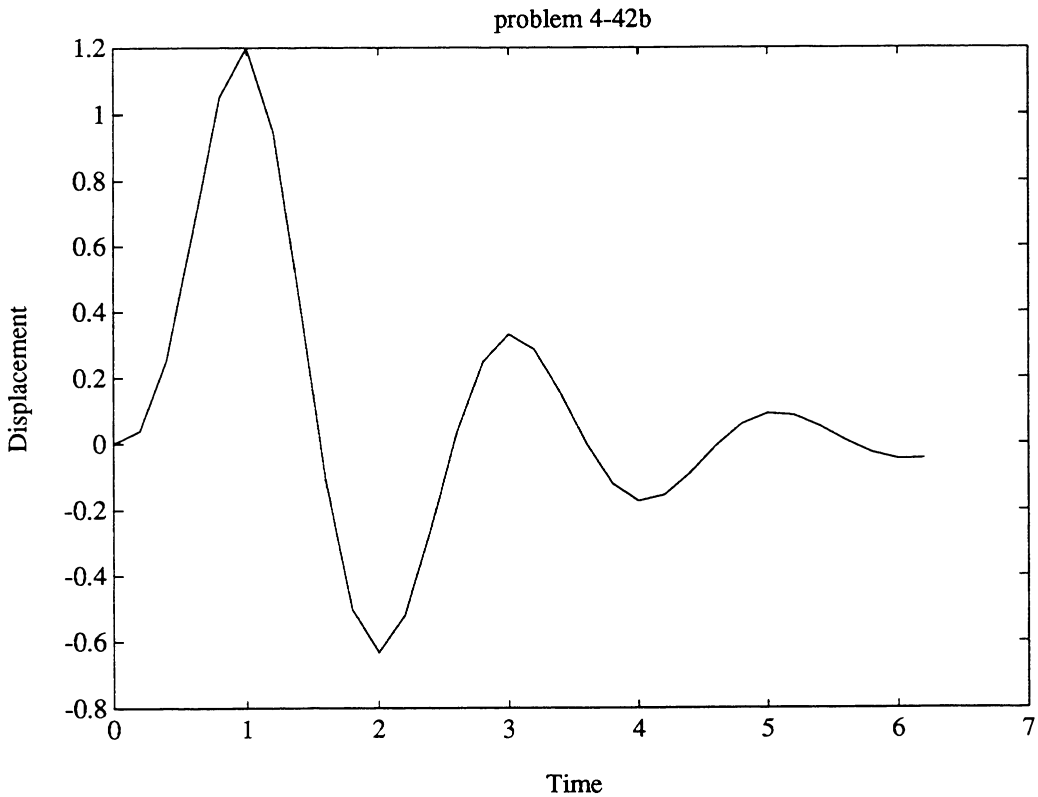
4-42

```
%This is the function for 4-42
function [force] = f(t)
 if t<.2
force=50*t;
elseif t<.4
force=10;
elseif t<.8
force=25*(t-.8);
else
force=0;
end
end
```



4-42 cont

```
%This is the function for 4-42b
function [force] = f(t)
 if t<1
force=10*sin(pi*t);
else
force=0;
end
end
```



4-43

The stiffness of the crane boom is represented by k_c, measured from the extended straight line.

$$\frac{W}{g}\ddot{y} = k_c(x-y) - W$$

$$\ddot{y} + \left(\frac{k_c g}{W}\right) y = \left(\frac{k_c g}{W}\right) x - g \qquad , \quad \omega^2 = \left(\frac{k_c g}{W}\right)$$

$$\ddot{y} + \omega^2 y = \omega^2 Vt - g \qquad , \quad x = Vt$$

By L.T.

$$s^2 \bar{y}(s) - s\,y(0) - \dot{y}(0) + \omega^2 \bar{y}(s) = \frac{\omega^2 V}{s^2} - \frac{g}{s}$$

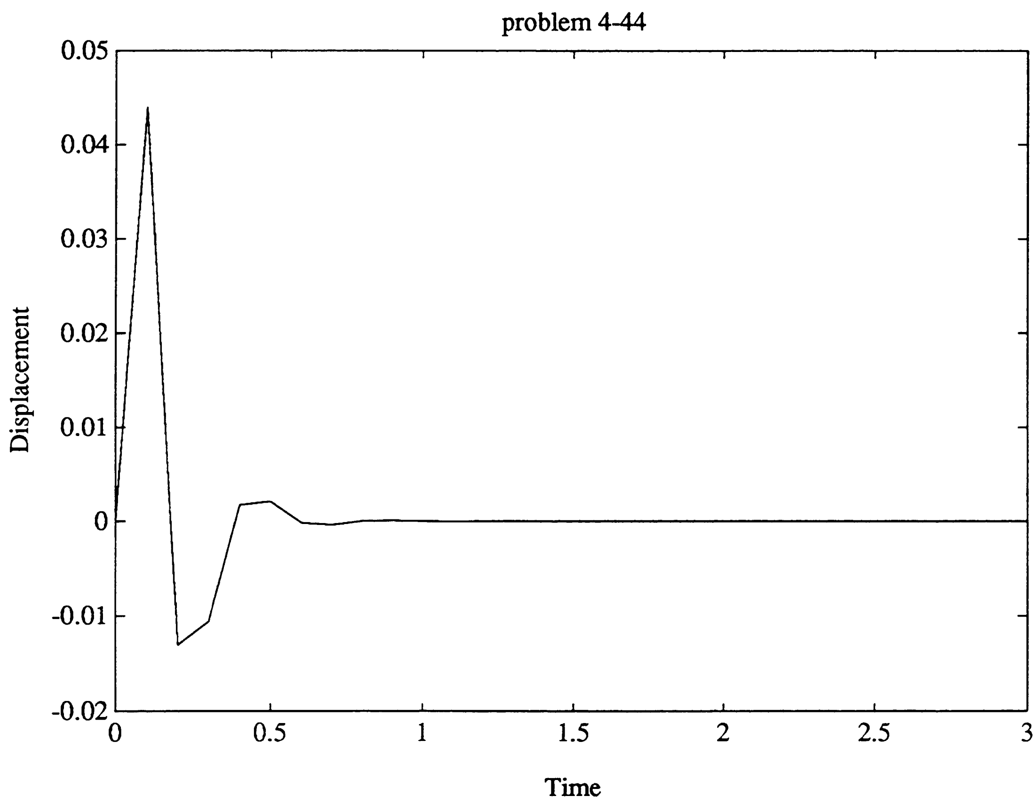
$$\bar{y}(s) = \frac{s\,y(0)}{s^2+\omega^2} + \frac{\dot{y}(0)}{s^2+\omega^2} + \frac{\omega^2 V}{s^2(s^2+\omega^2)} - \frac{g}{s(s^2+\omega^2)}$$

Inverse L.T.

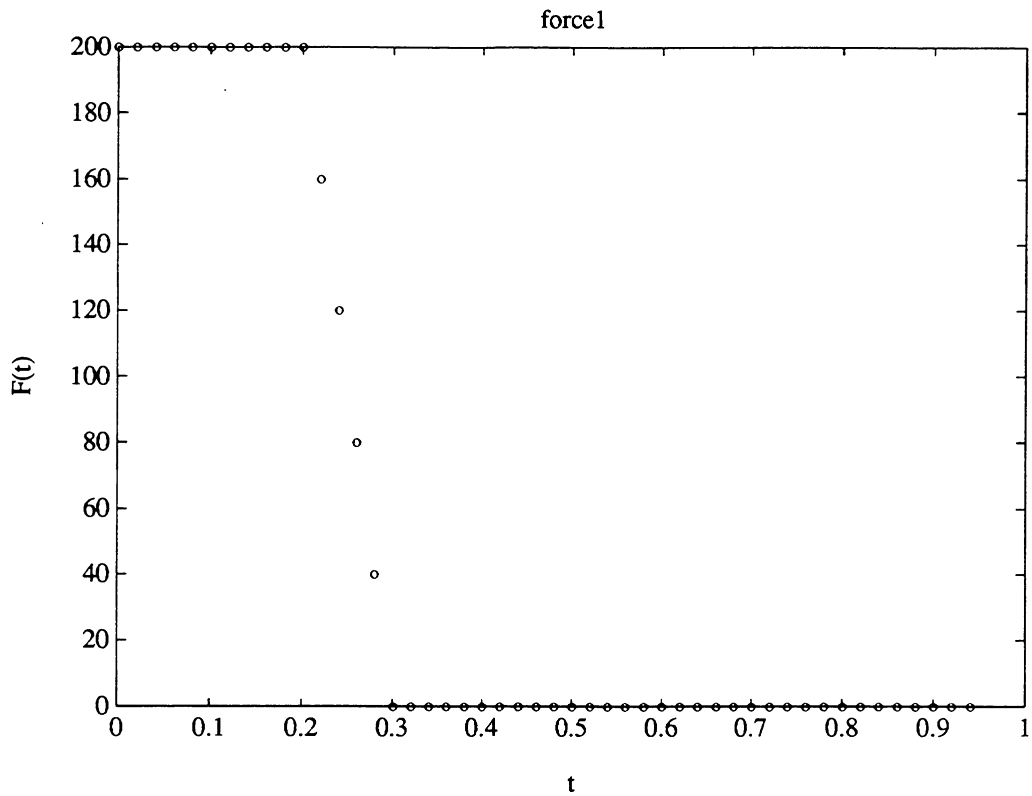
$$y(t) = y(0)\cos\omega t + \frac{\dot{y}(0)}{\omega}\sin\omega t + \frac{V}{\omega}(\omega t - \sin\omega t) - \frac{g}{\omega^2}(1-\cos\omega t)$$

4-44

```
%This is the function for 4-44
function [force] = f(t)
 if t<.1
force=100;
elseif t<.2
force=1000*t-200;
else
force =0;
end
end
```



4-45

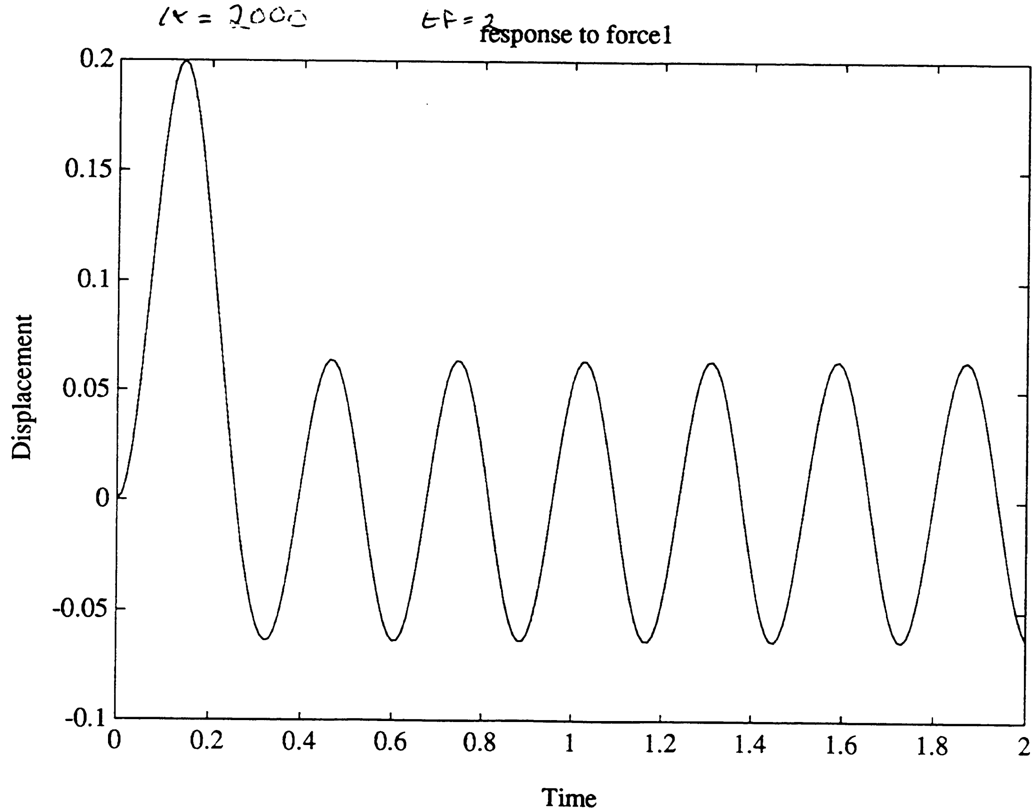


use program runga.m

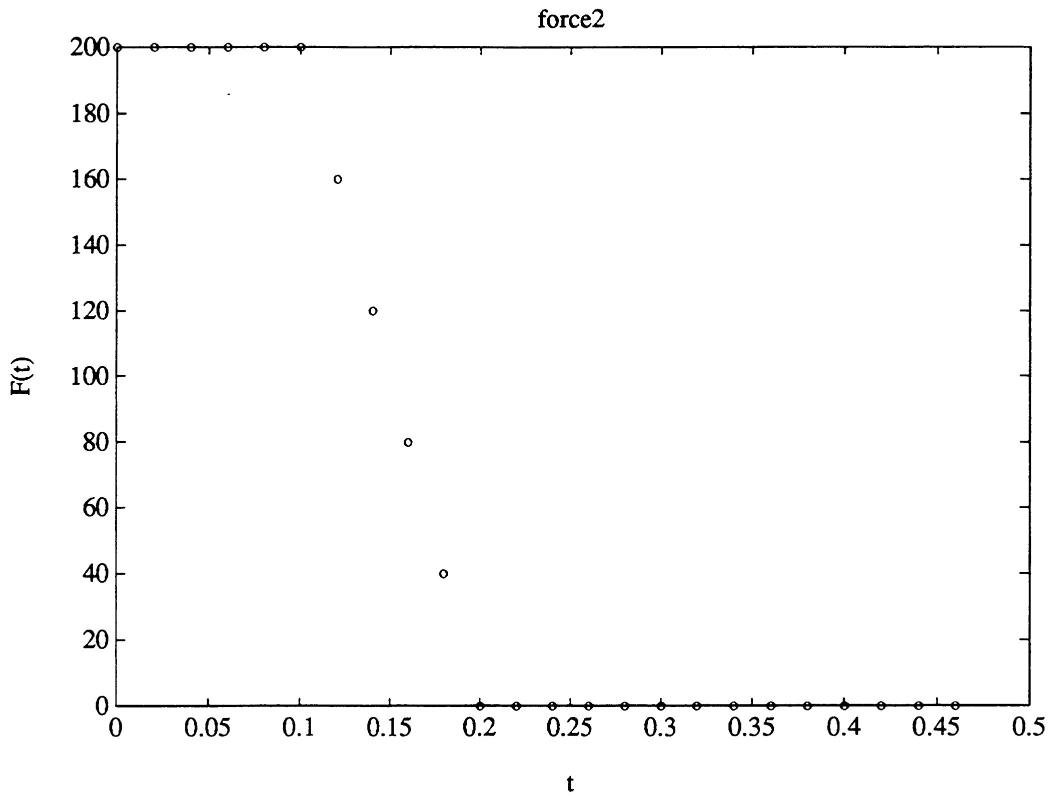
$m = 4$ $\quad dt = .01$ $\quad x_0 = \dot{x}_0 = 0$

$c = 0$ $\quad t_0 = 0$

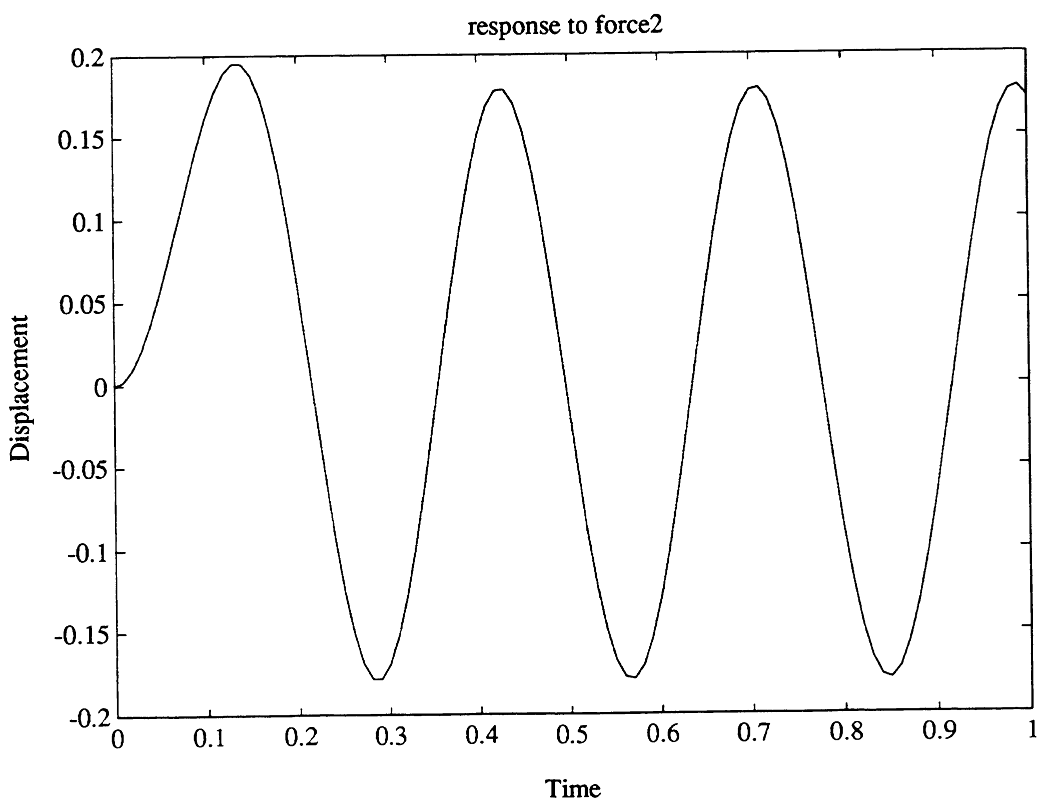
$k = 2000$ $\quad tf = 2$



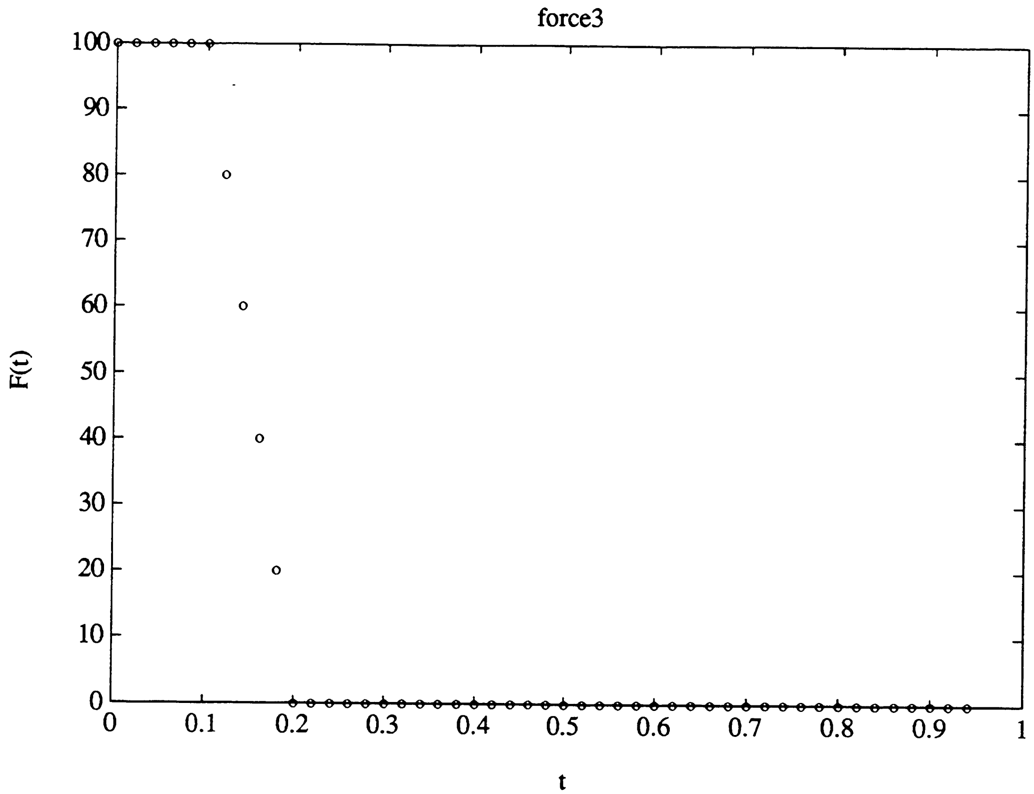
4-45 cont.



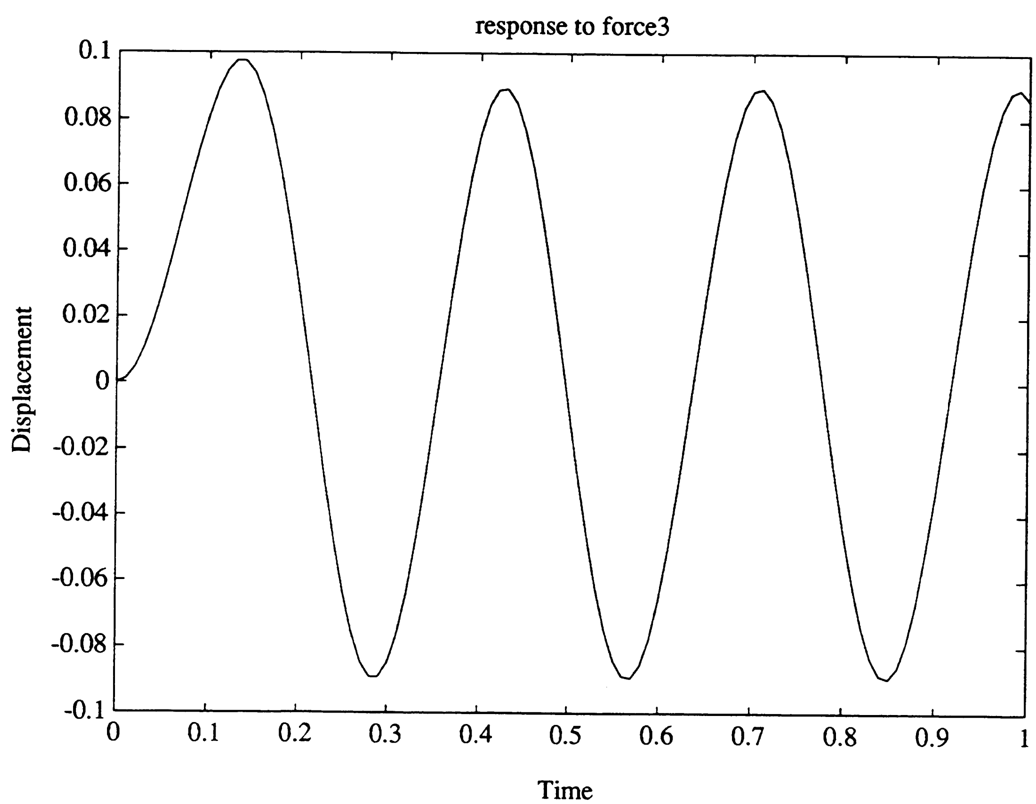
Same input as previous forcing



4-45 Cont



force3
F(t)
100
90
80
70
60
50
40
30
20
10
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t

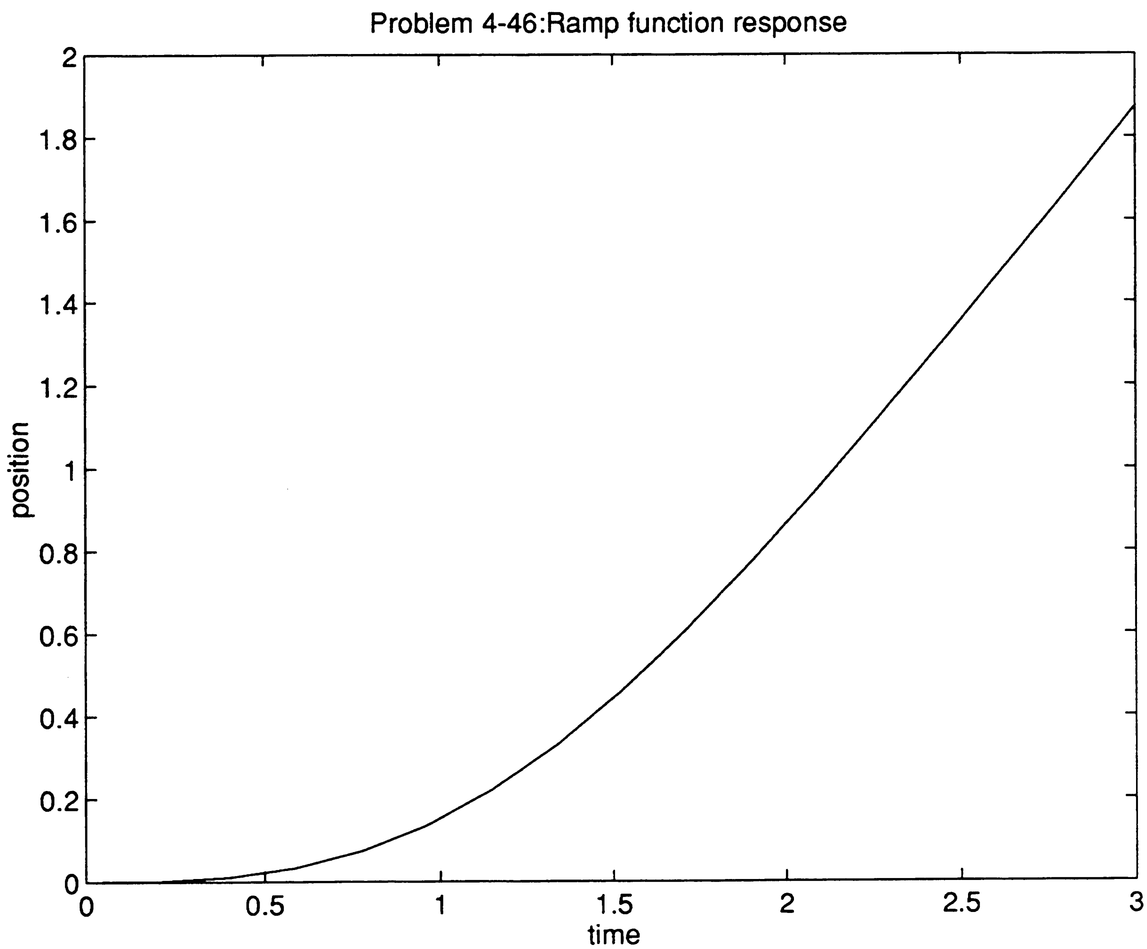


response to force3
Displacement
0.1
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
-0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Time

4-46

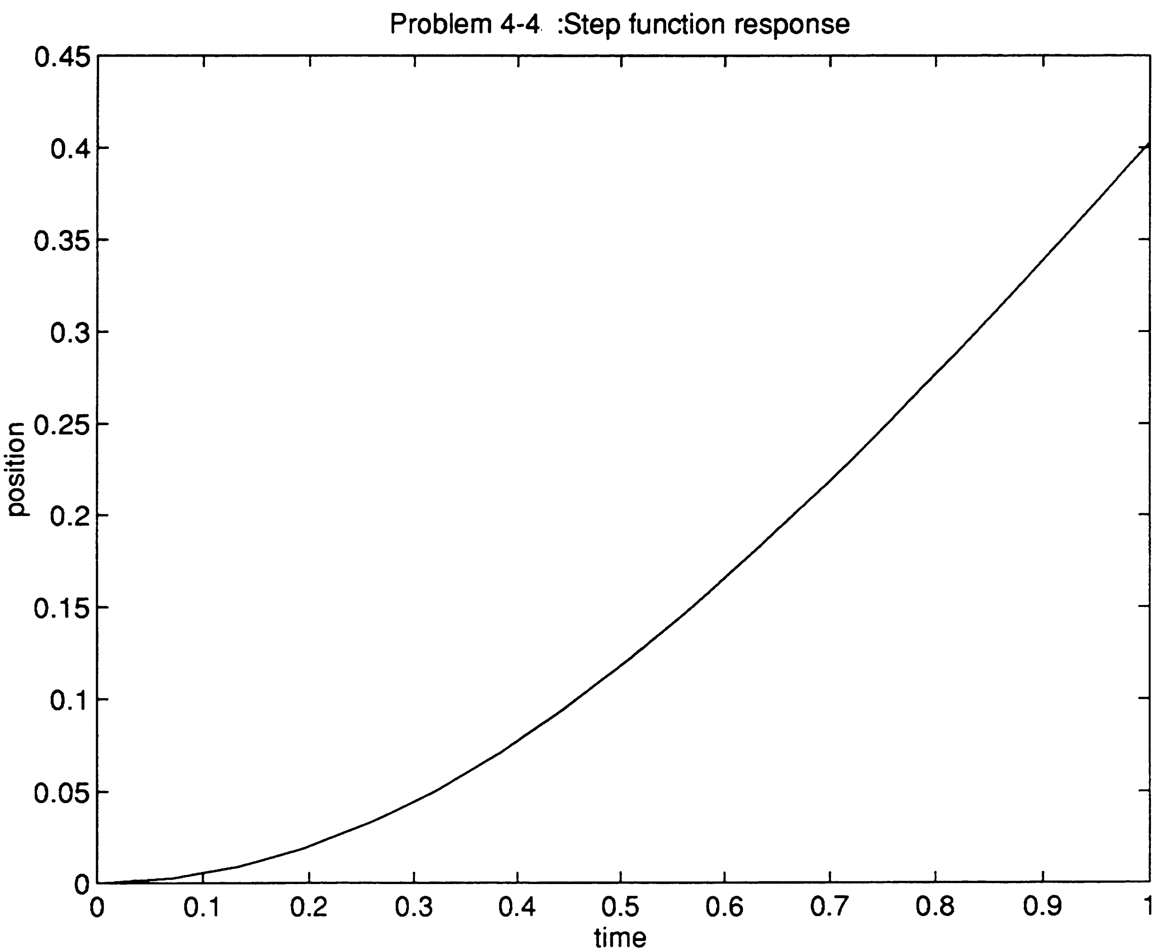
```
clear
global m k a step r
'Problem 4-46 Solution'
x0(1)=0;x0(2)=0;t0=0;
m=input('Enter the value of the mass (kg) : ')
k=input('Enter the value of the spring stiffness : ')
a=input('Enter the value of the damping coefficient a : ')
choice=input('Enter your choice of the forcing function function step=1 and
ramp=2 : ')
if choice==1
   step=input('Enter the value of the step function : ')
   tf=input('Enter the value of the final time : ')
   [t,x]=ode45('p449s',t0,tf,x0);
   plot(t,x(:,1));xlabel('time');ylabel('position');title('Problem 4-4 :Step
function response')
end
if choice==2
   r=input('Enter the slope of the ramp function : ')
   tf=input('Enter the value of the final time : ')
   [t,x]=ode45('p449r',t0,tf,x0);
   plot(t,x(:,1));xlabel('time');ylabel('position');title('Problem 4-4 :Ramp
function response')
end
function xdot=p449r(t,x)
global m k a r
if x(2)>0 | x(2)==0
   xdot=[x(2);
         (1/m)*(-a*x(2)^2-k*x(1)+r*t)];
else
   xdot=[x(2);
         (1/m)*(a*x(2)^2-k*x(1)+r*t)];
end


function xdot=p449s(t,x)
global m k a step
if x(2)>0 | x(2)==0
   xdot=[x(2);
         (1/m)*(-a*x(2)^2-k*x(1)+step)];
else
   xdot=[x(2);
         (1/m)*(a*x(2)^2-k*x(1)+step)];
end
```

4-46cont



Problem 4-46:Ramp function response
position
time
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
0.5
1.5
2.5
3

4-46 cont



Problem 4-4 :Step function response
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
position
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
time

4-47

```
clear
global m k c mu step r
'Problem 4-47 Solution'
x0(1)=0;x0(2)=0;t0=0;
m=input('Enter the value of the mass (kg) : ')
c=input('Enter the value of the damping coefficient (N.s/m) : ')
k=input('Enter the value of the spring stiffness (N/m): ')
mu=input('Enter the value of mu (N/m^3) : ')
choice=input('Enter your choice of the forcing function function step=1 and ramp=2 : ')
if choice==1
   step=input('Enter the value of the step function : ')
   tf=input('Enter the value of the final time : ')
    [t,x]=ode45('p447s',t0,tf,x0);
     plot(t,x(:,1));xlabel('time');ylabel('position');title('Problem 4-47:Step function
response')
end
if choice==2
   r=input('Enter the slope of the ramp function : ')
   tf=input('Enter the value of the final time : ')
    [t,x]=ode45('p447r',t0,tf,x0);
     plot(t,x(:,1));xlabel('time');ylabel('position');title('Problem 4-47:Ramp function
response')
end
```

```
function xdot=p447r(t,x)
global m k c mu r
   xdot=[x(2);
          (1/m)*(-c*x(2)-k*x(1)+mu*x(1)^3+r*t)];
```

```
function xdot=p447s(t,x)
global m k c mu step
   xdot=[x(2);
          (1/m)*(-c*x(2)-k*x(1)+mu*x(1)^3+step)];
```

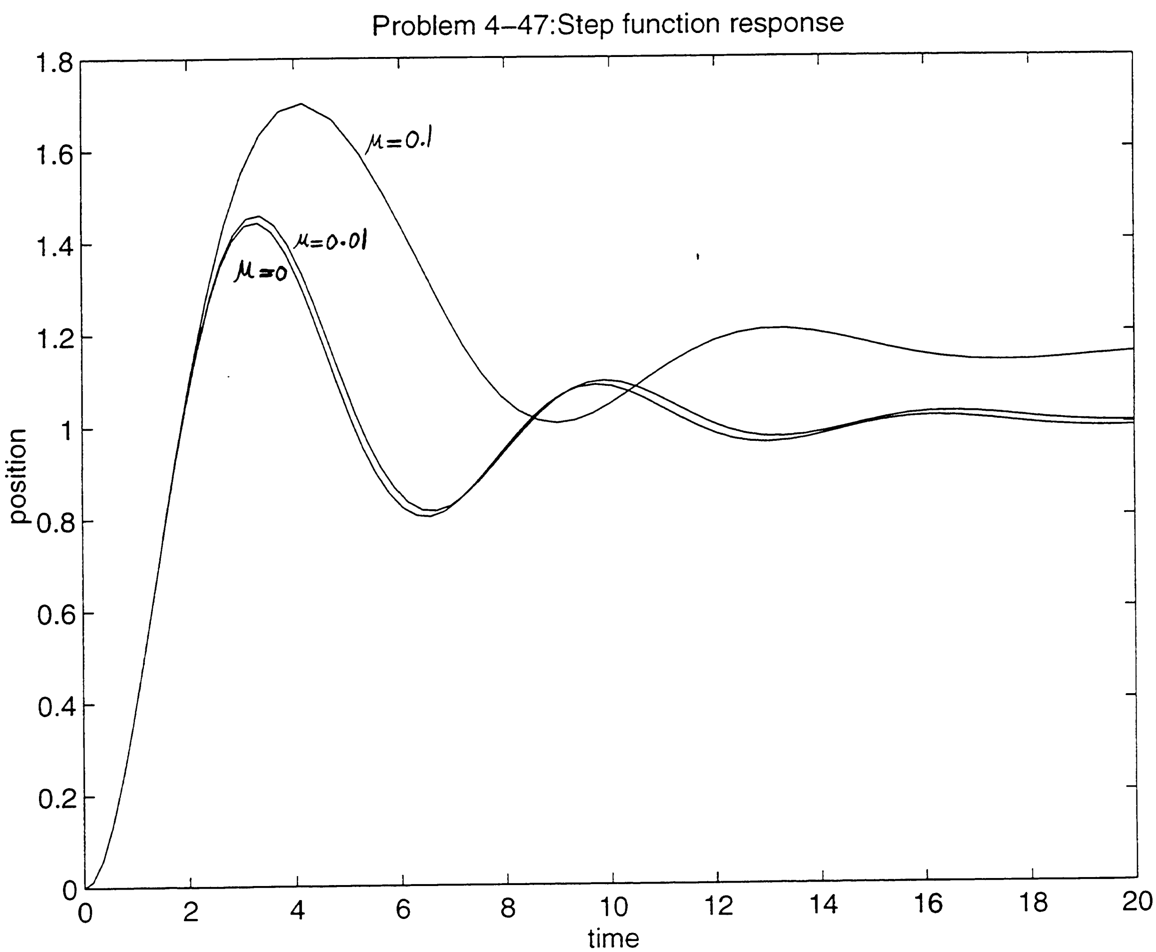
4-47

```
clear
global m k c mu step r
'Problem 4-47 Solution'
x0(1)=0;x0(2)=0;t0=0;
m=input('Enter the value of the mass (kg) : ')
c=input('Enter the value of the damping coefficient (N.s/m) : ')
k=input('Enter the value of the spring stiffness (N/m): ')
mu=input('Enter the value of mu (N/m^3) : ')
choice=input('Enter your choice of the forcing function function step=1 and ramp=2 : ')
if choice==1
   step=input('Enter the value of the step function : ')
   tf=input('Enter the value of the final time : ')
    [t,x]=ode45('p447s',t0,tf,x0);
     plot(t,x(:,1));xlabel('time');ylabel('position');title('Problem 4-47:Step function
response')
end
if choice==2
   r=input('Enter the slope of the ramp function : ')
   tf=input('Enter the value of the final time : ')
    [t,x]=ode45('p447r',t0,tf,x0);
     plot(t,x(:,1));xlabel('time');ylabel('position');title('Problem 4-47:Ramp function
response')
end
```

```
function xdot=p447r(t,x)
global m k c mu r
   xdot=[x(2);
          (1/m)*(-c*x(2)-k*x(1)+mu*x(1)^3+r*t)];
```

```
function xdot=p447s(t,x)
global m k c mu step
   xdot=[x(2);
          (1/m)*(-c*x(2)-k*x(1)+mu*x(1)^3+step)];
```

4-47 cont



Problem 4–47:Step function response
position
time
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
0
2
4
6
8
10
12
14
16
18
20
$\mu=0.1$
$\mu=0.01$
$\mu=0$

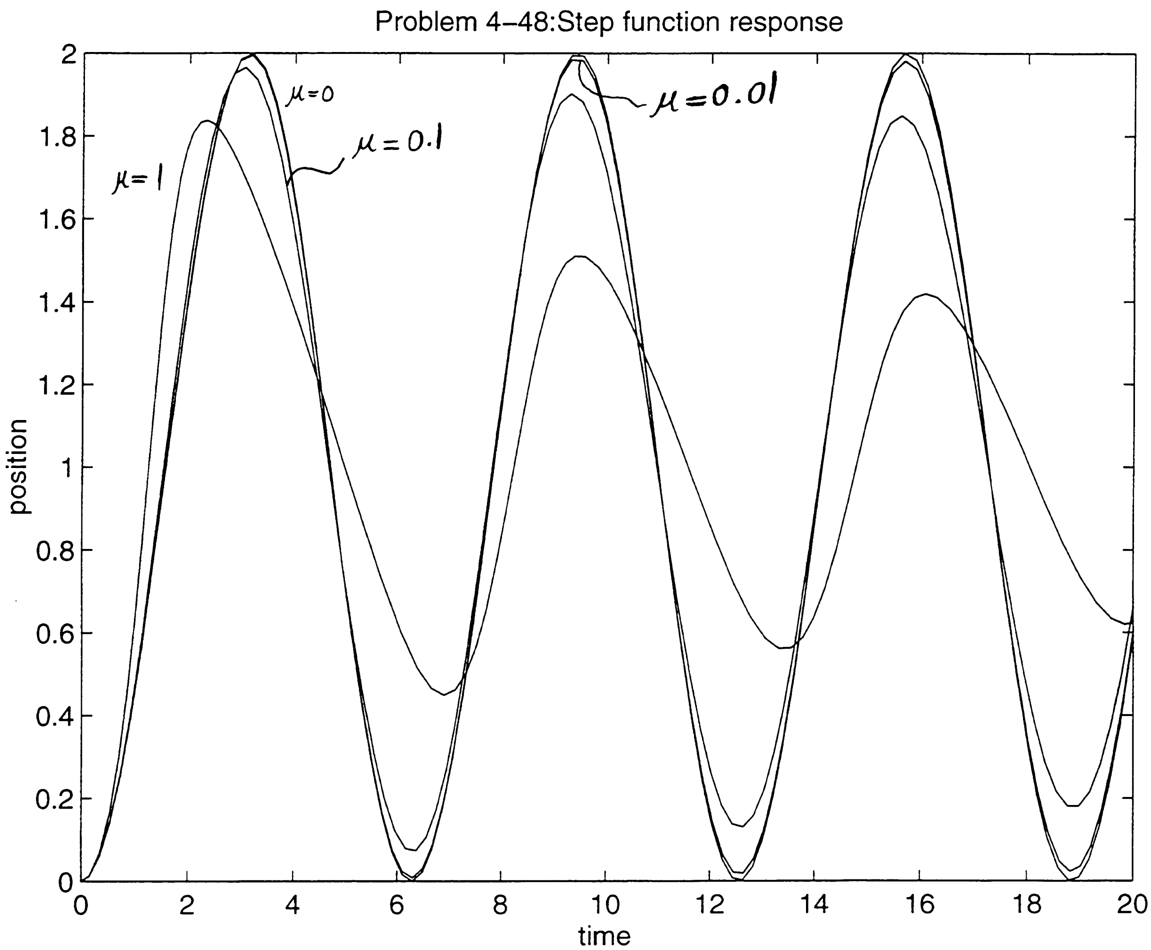
4-48

```
clear
global mu step r
'Problem 4-48 Solution'
x0(1)=0;x0(2)=0;t0=0;
mu=input('Enter the value mu : ')
choice=input('Enter your choice of the forcing function function step=1 and ramp=2 : ')
if choice==1
   step=input('Enter the value of the step function : ')
   tf=input('Enter the value of the final time : ')
    [t,x]=ode45('p448s',t0,tf,x0);
     plot(t,x(:,1));xlabel('time');ylabel('position');title('Problem 4-48:Step function
response')
end
if choice==2
   r=input('Enter the slope of the ramp function : ')
   tf=input('Enter the value of the final time : ')
    [t,x]=ode45('p448r',t0,tf,x0);
     plot(t,x(:,1));xlabel('time');ylabel('position');title('Problem 4-48:Ramp function
response')
end
```

```
function xdot=p448s(t,x)
global mu step
   xdot=[x(2);
         mu*x(2)*(1-x(1)^2)-x(1)+step];
```

```
function xdot=p448r(t,x)
global mu r
   xdot=[x(2);
         mu*x(2)*(1-x(1)^2)-x(1)+r*t];
```

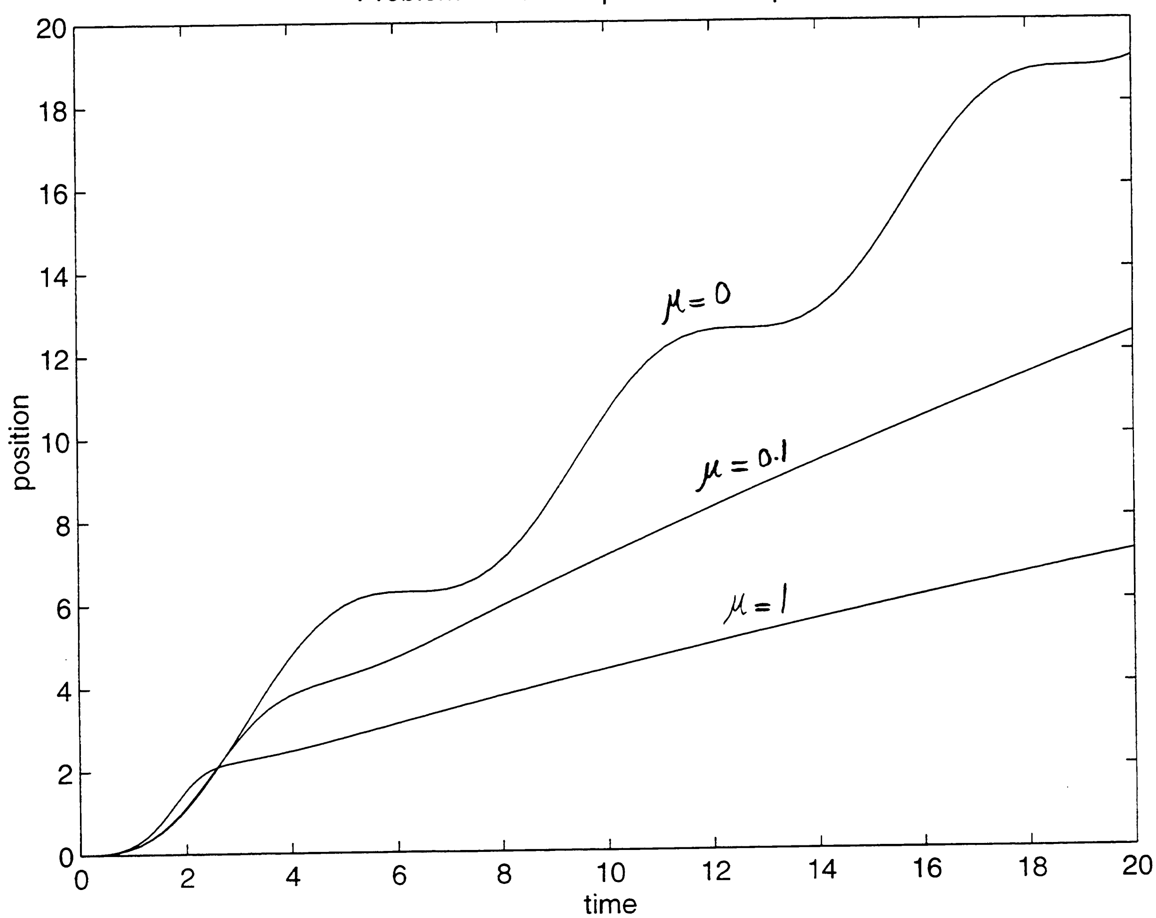
4-48 cont



Problem 4–48:Step function response
position
time
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
0
2
4
6
8
10
12
14
16
18
20
$\mu=0$
$\mu=0.1$
$\mu=1$
$\mu=0.01$

4-48 cont

Problem 4–48:Ramp function response



5-1

$$m\ddot{x}_1 = -kx_1 + k(x_2 - x_1)$$
$$m\ddot{x}_2 = -k(x_2 - x_1)$$

origin of x_1 & x_2 = equilib position

$$\left.\begin{aligned}(2-\tfrac{m\omega^2}{k})X_1 &= X_2 \\ X_1 &= (1-\tfrac{m\omega^2}{k})X_2\end{aligned}\right\} \text{let } \lambda = \frac{m\omega^2}{k}$$

characteristic eq. $\lambda^2 - 3\lambda + 1 = 0$

$$\lambda = \begin{cases} 0.382 = \dfrac{m\omega_1^2}{k} \\ 2.618 = \dfrac{m\omega_2^2}{k}\end{cases} \qquad \begin{aligned}(X_1/X_2)_1 &= 1-\lambda_1 = 0.614 \\ (X_1/X_2)_2 &= 1-\lambda_2 = -1.618\end{aligned}$$

5-2 & 3

$$\begin{bmatrix} m & 0 \\ 0 & m\end{bmatrix}\begin{Bmatrix}\ddot{x}_1 \\ \ddot{x}_2\end{Bmatrix} + \begin{bmatrix}(k+nk) & -nk \\ -nk & (k+nk)\end{bmatrix}\begin{Bmatrix}x_1 \\ x_2\end{Bmatrix} = \{0\}$$

Let $\lambda = m\omega^2/k$

$$\begin{vmatrix}(1+n-\lambda) & -n \\ -n & (1+n-\lambda)\end{vmatrix} = 0 \qquad \begin{aligned}&\lambda^2 - 2(1+n)\lambda + (1+2n) = 0 \\ &\lambda = (1+n) \pm n, \quad X_1/X_2 = \frac{1+n-\lambda}{n}\end{aligned}$$

For $n = 1$.

$$\lambda_1 = 1 = \frac{m\omega_1^2}{k} \qquad (X_1/X_2)_1 = 1$$
$$\lambda_2 = 3 = \frac{m\omega_2^2}{k} \qquad (X_1/X_2)_2 = -1$$

5-4

$$m\begin{bmatrix}3 & 0 \\ 0 & 1\end{bmatrix}\begin{Bmatrix}\ddot{x}_1 \\ \ddot{x}_2\end{Bmatrix} + k\begin{bmatrix}2 & -1 \\ -1 & 4\end{bmatrix}\begin{Bmatrix}x_1 \\ x_2\end{Bmatrix} = \{0\}$$

$\lambda = \dfrac{m\omega^2}{k}$

$$\begin{vmatrix}(2-3\lambda) & -1 \\ -1 & (4-\lambda)\end{vmatrix} = 0 \qquad \begin{aligned}&\lambda^2 - \tfrac{14}{3}\lambda + \tfrac{7}{3} = 0 \\ &\lambda = \tfrac{14}{6} \pm \sqrt{(\tfrac{14}{6})^2 - \tfrac{7}{3}} = \begin{cases}0.570 = \dfrac{m\omega_1^2}{k} \\ 4.096 = \dfrac{m\omega_2^2}{k}\end{cases}\end{aligned}$$

$$\frac{X_1}{X_2} = (4-\lambda) \qquad \left(\frac{X_1}{X_2}\right)_1 = 3.43 \qquad \left(\frac{X_1}{X_2}\right)_2 = -0.096$$

5-5

$$J_2\begin{bmatrix}2 & 0 \\ 0 & 1\end{bmatrix}\begin{Bmatrix}\ddot{\theta}_1 \\ \ddot{\theta}_2\end{Bmatrix} + K\begin{bmatrix}2 & -1 \\ -1 & 1\end{bmatrix}\begin{Bmatrix}\theta_1 \\ \theta_2\end{Bmatrix} = \{0\} \qquad \lambda = \frac{\omega^2 J_2}{K}$$

$$\begin{vmatrix}(2-2\lambda) & -1 \\ -1 & (1-\lambda)\end{vmatrix} = 0 \qquad \begin{aligned}&2(1-\lambda)^2 - 1 = 0 \\ &(1-\lambda) = \pm 1/\sqrt{2} \qquad \lambda = 1 \mp 1/\sqrt{2}\end{aligned}$$

$$\lambda_{1,2} = \begin{cases}.293 = \dfrac{J_2\omega_1^2}{K} \\ 1.707 = \dfrac{J_2\omega_2^2}{K}\end{cases} \qquad \left(\frac{\theta_1}{\theta_2}\right)_1 = 0.707 \qquad \left(\frac{\theta_1}{\theta_2}\right)_2 = -0.707$$

5-1

$$m\ddot{x}_1 = -kx_1 + k(x_2 - x_1)$$
$$m\ddot{x}_2 = -k(x_2 - x_1)$$

origin of x_1 & x_2 = equilib position

$$\left.\begin{aligned}(2 - \tfrac{m\omega^2}{k})X_1 &= X_2 \\ X_1 &= (1 - \tfrac{m\omega^2}{k})X_2\end{aligned}\right\} \text{let } \lambda = \frac{m\omega^2}{k}$$

characteristic eq. $\lambda^2 - 3\lambda + 1 = 0$

$$\lambda = \begin{cases} 0.382 = \dfrac{m\omega_1^2}{k} \\ 2.618 = \dfrac{m\omega_2^2}{k} \end{cases} \qquad \begin{aligned}(X_1/X_2)_1 &= 1-\lambda_1 = 0.614 \\ (X_1/X_2)_2 &= 1-\lambda_2 = -1.618\end{aligned}$$

5-2 & 3

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (k+nk) & -nk \\ -nk & (k+nk) \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \{0\}$$

Let $\lambda = m\omega^2/k$

$$\begin{vmatrix} (1+n-\lambda) & -n \\ -n & (1+n-\lambda) \end{vmatrix} = 0 \qquad \begin{aligned}&\lambda^2 - 2(1+n)\lambda + (1+2n) = 0 \\ &\lambda = (1+n) \pm n, \quad X_1/X_2 = \frac{1+n-\lambda}{n}\end{aligned}$$

For $n = 1$.
$$\lambda_1 = 1 = \frac{m\omega_1^2}{k} \qquad (X_1/X_2)_1 = 1$$
$$\lambda_2 = 3 = \frac{m\omega_2^2}{k} \qquad (X_1/X_2)_2 = -1$$

5-4

$$m\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + k\begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \{0\}$$

$\lambda = \dfrac{m\omega^2}{k}$

$$\begin{vmatrix} (2-3\lambda) & -1 \\ -1 & (4-\lambda) \end{vmatrix} = 0 \qquad \lambda^2 - \frac{14}{3}\lambda + \frac{7}{3} = 0$$

$$\lambda = \frac{14}{6} \pm \sqrt{\left(\frac{14}{6}\right)^2 - \frac{7}{3}} = \begin{cases} 0.570 = \dfrac{m\omega_1^2}{k} \\ 4.096 = \dfrac{m\omega_2^2}{k} \end{cases}$$

$$\frac{X_1}{X_2} = (4-\lambda) \qquad \left(\frac{X_1}{X_2}\right)_1 = 3.43 \qquad \left(\frac{X_1}{X_2}\right)_2 = -0.096$$

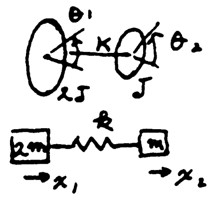
5-5

$$J_2\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + K\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \{0\} \qquad \lambda = \frac{\omega^2 J_2}{K}$$

$$\begin{vmatrix} (2-2\lambda) & -1 \\ -1 & (1-\lambda) \end{vmatrix} = 0 \qquad \begin{aligned}2(1-\lambda)^2 - 1 &= 0 \\ (1-\lambda) &= \pm 1/\sqrt{2}\end{aligned} \qquad \lambda = 1 \mp 1/\sqrt{2}$$

$$\lambda_{1,2} = \begin{cases} .293 = \dfrac{J_2\omega_1^2}{K} \\ 1.707 = \dfrac{J_2\omega_2^2}{K} \end{cases} \qquad \left(\frac{\theta_1}{\theta_2}\right)_1 = 0.707 \qquad \left(\frac{\theta_1}{\theta_2}\right)_2 = -0.707$$

5-6



for $k_1 = 0$

$$\begin{vmatrix} (1-2\lambda) & -1 \\ -1 & (1-\lambda) \end{vmatrix} = 0 \qquad \lambda(2\lambda - 3) = 0$$

$$\therefore \lambda = 0, \quad \lambda = 1.5$$

$\uparrow$ free rotation

$$\frac{\theta_1}{\theta_2} = -\frac{1}{2}$$

In general

$$J_1 \ddot{\theta}_1 - K_2(\theta_2 - \theta_1) = 0$$

$$J_2 \ddot{\theta}_2 + K_2(\theta_2 - \theta_1) = 0$$

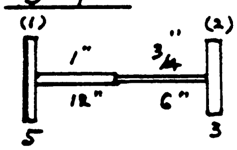
$$(\ddot{\theta}_2 - \ddot{\theta}_1) + \left(\frac{K_2}{J_2} + \frac{K_2}{J_1}\right)(\theta_2 - \theta_1) = 0$$

Let $\phi = \pm(\theta_2 - \theta_1)$

$$\ddot{\phi} + K_2\left(\frac{J_1 + J_2}{J_1 J_2}\right)\phi = 0$$

$$\omega_n = \sqrt{K_2\left(\frac{J_1 + J_2}{J_1 J_2}\right)}$$

5-7



$$K_1 = \frac{GI_p}{\ell} = \frac{(11.5 \times 10^6)}{12''}\frac{\pi 1^4}{32} = 0.0941 \times 10^6$$

$$K_2 = \frac{(11.5 \times 10^6)}{6''}\frac{\pi(\frac{3}{4})^4}{32} = 0.0595 \times 10^6$$

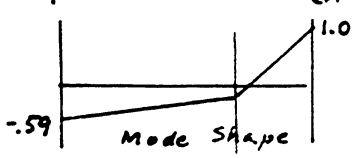
$$K_{eff} = \frac{K_1 K_2}{K_1 + K_2} = 0.0365 \times 10^6$$

$$\omega_n = \sqrt{K_{eff}\left(\frac{J_1 + J_2}{J_1 J_2}\right)} = \sqrt{\frac{8}{15} \times .0365 \times 10^6} = 139.4 \text{ rad/s.}$$

characteristic eq.

$$\begin{vmatrix} (1 - \frac{\omega_n^2 J_1}{K_{eff}}) & -1 \\ -1 & (1 - \frac{\omega_n^2 J_2}{K_{eff}}) \end{vmatrix} = 0 \qquad \left(\frac{\theta_1}{\theta_2}\right) = \left(1 - \frac{\omega^2 J_2}{K_{eff}}\right) = 1 - 139.4^2 \frac{3}{.0365 \times 10^6}$$

$$= 1 - 1.5979 = -0.5979$$

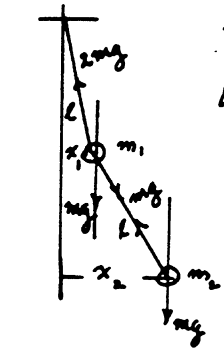


Twist in each shaft $\propto$ to $\frac{1}{K_i}$

5-8 Total stiffness between cars = 16,000 #/in. System has node at middle of spring $\therefore$ oscillates as two single deg. freed. system with each spring of $2k = 32{,}000$ #/in and mass m

$$\omega_n = \sqrt{\frac{2k}{m}} = \sqrt{k\left(\frac{m_1 + m_2}{m_1 m_2}\right)} = \sqrt{\frac{2 \times 16{,}000 \times 386}{50{,}000}} = 15.72 \text{ rad/s.}$$

5-9



For small amplitudes angles are $\frac{x_1}{\ell}$ and $\frac{x_2-x_1}{\ell}$.
Tensions are approx $2mg$ and mg.
Use $\sum F_x$

$$m\ddot{x}_1 = -2mg\left(\frac{x_1}{\ell}\right) + mg\left(\frac{x_2-x_1}{\ell}\right)$$

$$m\ddot{x}_2 = -mg\left(\frac{x_2-x_1}{\ell}\right) \qquad \text{Let } \lambda = \frac{\omega^2 \ell}{g}$$

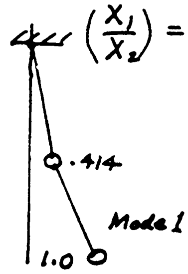
$$(3-\lambda)X_1 - X_2 = 0$$
$$-X_1 + (1-\lambda)X_2 = 0$$

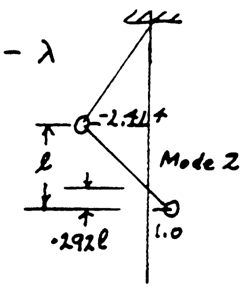
$$\lambda^2 - 4\lambda + 2 = 0\ , \qquad \lambda = 2 \pm \sqrt{2} = \begin{cases} 0.586 \\ 3.414 \end{cases}$$

$$\omega_n = \sqrt{(2\pm\sqrt{2})\,g/\ell} = \begin{cases} 0.764\sqrt{\frac{g}{\ell}} \\ 1.850\sqrt{\frac{g}{\ell}} \end{cases} \qquad \left(\frac{X_1}{X_2}\right) = 1-\lambda$$

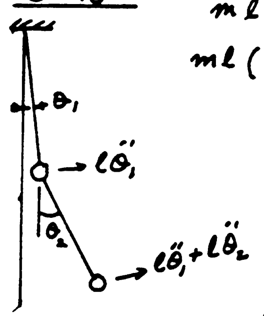
$$\left(\frac{X_1}{X_2}\right)_1 = 1-\lambda_1 = 0.414$$

$$\left(\frac{X_1}{X_2}\right)_2 = -2.414$$





5-10



$$m\ell\ddot{\theta}_1 = -2mg\,\theta_1 + mg\,\theta_2$$

$$m\ell(\ddot{\theta}_1 + \ddot{\theta}_2) = -mg\,\theta_2 \qquad \lambda = \frac{\omega^2 \ell}{g}$$

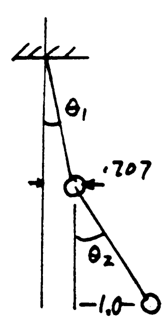
$$(2-\lambda)\theta_1 - \theta_2 = 0$$
$$-\lambda\theta_1 + (1-\lambda)\theta_2 = 0$$

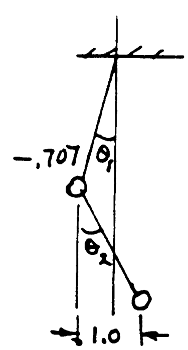
$$\lambda^2 - 4\lambda + 2 = 0 \qquad \text{Same as Prob 5-9}$$

$$\frac{\theta_1}{\theta_2} = \frac{1-\lambda}{\lambda}$$

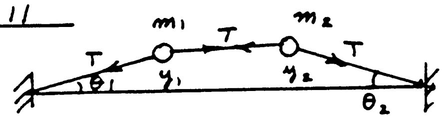
$$\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}_1 = \begin{Bmatrix} 0.707 \\ 1.00 \end{Bmatrix}$$

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}_2 = \begin{Bmatrix} -0.707 \\ 1.00 \end{Bmatrix}$$





5-11



$\theta_1 \cong y_1/\ell \qquad \theta_2 \cong y_2/\ell$

$$m_1 \ddot{y}_1 = -T\frac{y_1}{\ell} + T\left(\frac{y_2-y_1}{\ell}\right)$$

$$m_2 \ddot{y}_2 = -T\left(\frac{y_2-y_1}{\ell}\right) - T\frac{y_2}{\ell}$$

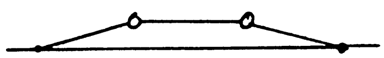
$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} + \frac{T}{\ell}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \{0\}$$

5-12 Let $\lambda = \omega^2 \frac{m\ell}{T}$ in Prob 5-11

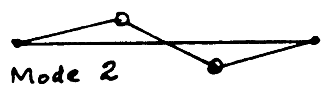
$$\begin{bmatrix} (2-\lambda) & -1 \\ -1 & (2-\lambda) \end{bmatrix}\begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} = \{0\} \qquad \lambda^2 - 4\lambda + 3 = 0$$

$$\lambda_1 = 1 \qquad \omega_1 = \sqrt{T/m\ell} \qquad \left(\frac{Y_1}{Y_2}\right)_1 = 1$$

$$\lambda_2 = 3 \qquad \omega_2 = \sqrt{3T/m\ell} \qquad \left(\frac{Y_1}{Y_2}\right)_2 = -1$$



Mode 1



Mode 2

5-13

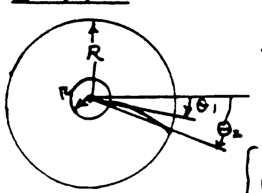
$$\begin{vmatrix} (2-2\lambda) & -1 \\ -1 & (2-\lambda) \end{vmatrix} = 0 \qquad 2\lambda^2 - 6\lambda + 3 = 0$$

$$\lambda = \frac{3}{2} \pm \sqrt{\left(\frac{3}{2}\right)^2 - 1.5}$$

$$\lambda_1 = 0.634 \qquad \omega_1 = 0.796\sqrt{T/m\ell} \qquad \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix}_1 = \begin{Bmatrix} 1.00 \\ 0.732 \end{Bmatrix}$$

$$\lambda_2 = 2.366 \qquad \omega_2 = 1.538\sqrt{T/m\ell} \qquad \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix}_2 = \begin{Bmatrix} 1.00 \\ -2.732 \end{Bmatrix}$$

5-14



$$\left.\begin{aligned} J_1\ddot{\theta}_1 &= -K_1\theta_1 + K_2(\theta_2-\theta_1) \\ J_2\ddot{\theta}_2 &= -K_2(\theta_2-\theta_1) \end{aligned}\right\} \quad K_2 = 4k_2R^2$$

Defl. of spring $=$

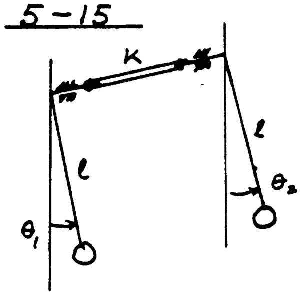
$$R(\theta_2-\theta_1) = \frac{\text{Force}}{k_2} = \frac{F}{k_2}$$

$$\text{Torque} = RF = k_2R^2(\theta_2-\theta_1)$$

$$\begin{bmatrix} \left(\frac{K_1+K_2}{J_1} - \omega^2\right) & -\frac{K_2}{J_1} \\ -\frac{K_2}{J_2} & \left(\frac{K_2}{J_2} - \omega^2\right) \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \{0\}$$

$$\omega^4 - \left[\frac{K_1}{J_1} + \frac{K_2}{J_1} + \frac{K_2}{J_2}\right]\omega^2 + \frac{K_1}{J_1}\frac{K_2}{J_2} = 0$$

5-15



$$ml^2\ddot{\theta}_1 = -mgl\theta_1 + k(\theta_2 - \theta_1)$$
$$ml^2\ddot{\theta}_2 = -mgl\theta_2 - k(\theta_2 - \theta_1)$$

$$\begin{bmatrix} \left(\frac{g}{l} + \frac{k}{ml^2} - \omega^2\right) & -\frac{k}{ml^2} \\ -\frac{k}{ml^2} & \left(\frac{g}{l} + \frac{k}{ml^2} - \omega^2\right) \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix}$$

$$\omega^4 - 2\left(\frac{g}{l} + \frac{k}{ml^2}\right)\omega^2 + \frac{g}{l}\left(\frac{g}{l} + 2\frac{k}{ml^2}\right) = 0$$

$$\omega^2 = \frac{g}{l} + \frac{k}{ml^2}(1 \pm 1) \qquad \frac{\theta_1}{\theta_2} = \frac{\frac{g}{l} + \frac{k}{ml^2} - \omega^2)}{k/ml^2}$$

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \qquad \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

For $\left\{\begin{array}{l} l = 19.3'' \\ mg = 3.86 \text{ lb} \\ k = 2.0 \text{ lb.in/rad} \end{array}\right\}$ $\quad \omega_1 = \sqrt{\frac{g}{l}} = 4.4721$

$$\omega_2 = \sqrt{\frac{g}{l} + \frac{2k}{ml^2}} = \sqrt{20 + 1.0739} = 4.5906$$

For $\theta_1(0) = 0$, $\theta_2(0) = \theta_0$, $\dot{\theta}_1(0) = \dot{\theta}_2(0) = 0$

$$\theta_1 = \frac{\theta_0}{2}\cos\omega_1 t - \frac{\theta_0}{2}\cos\omega_2 t = \theta_0 \sin\tfrac{1}{2}(\omega_2 - \omega_1)t \cdot \sin\tfrac{1}{2}(\omega_1 + \omega_2)t$$
$$\theta_2 = \frac{\theta_0}{2}\cos\omega_1 t + \frac{\theta_0}{2}\cos\omega_2 t = \theta_0 \cos\tfrac{1}{2}(\omega_2 - \omega_1)t \cdot \cos\tfrac{1}{2}(\omega_1 + \omega_2)t$$

$$\left.\begin{array}{l} \theta_1 = \theta_0 \sin 0.0593t \cdot \sin 4.5314t \\ \theta_2 = \theta_0 \cos 0.0593t \cdot \cos 4.5314t \end{array}\right\}$$

Beat period $\tau_b = \frac{\pi}{.0593}$ $= 52.978$ sec.

5-16

From Prob. 5-4 $\quad \lambda = \left\{\begin{array}{l} 0.5691 \\ 4.0972 \end{array}\right.$ $\quad (X_1/X_2)_1 = 3.4309$, $\quad (X_1/X_2)_2 = -0.0972$

For $x_1(0) = A$, $x_2(0) = 0$

$\dot{x}_1(0) = \dot{x}_2(0) = 0$

$$x_1 = 3.4309\, C_1 \cos\omega_1 t - 0.0972\, C_2 \cos\omega_2 t$$
$$x_2 = 1.0000\, C_1 \cos\omega_1 t + 1.0000\, C_2 \cos\omega_2 t$$

i.e. Mode 1 is multiplied by C_1 and mode 2 by C_2

At $t = 0$

$$A = 3.4309\, C_1 - 0.0972\, C_2$$
$$0 = C_1 + C_2 \qquad \therefore \quad C_1 = -C_2$$

5-16 Cont.

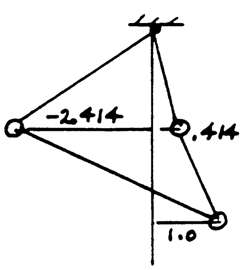
$$A = -3.4309\, C_2 - 0.0972\, C_2 = -3.5281\, C_2$$

$$\therefore C_2 = -0.2834 A\,, \qquad C_1 = 0.2834 A$$

$$x_1 = 0.9724 A \cos\omega_1 t + 0.0276 A \cos\omega_2 t$$

$$x_2 = 0.2834 A \cos\omega_1 t - 0.2834 A \cos\omega_2 t$$

5-17 Add A times mode 1 and B times mode 2



$$x_1 = 0.414 A \cos\omega_1 t - 2.414 B \cos\omega_2 t$$

$$x_2 = 1.00 A \cos\omega_1 t + 1.00 B \cos\omega_2 t$$

At $t = 0$ $\quad x_1(0) = x_2(0) = X$

$$X = 0.414 A - 2.414 B$$

$$X = 1 A + 1 B \qquad \therefore A = X - B$$

$$X = 0.414(X - B) - 2.414 B \qquad \therefore B = -0.2072 X$$

$$A = 1.2072 X$$

$$x_1 = X\left\{0.4998 \cos\omega_1 t + 0.5002 \cos\omega_2 t\right\}$$

$$x_2 = X\left\{1.2072 \cos\omega_1 t - 0.2072 \cos\omega_2 t\right\}$$

5-18 From Prob 5-1 $(X_1/X_2)_1 = 0.614$, $(X_1/X_2)_2 = -1.618$

$x_1(0) = \dot{x}_1(0) = x_2(0) = 0\,, \quad \dot{x}_2(0) = V$

The 4 initial conditions require 4 arbitrary constants. We choose A times mode 1, B times mode 2 and phase angles φ_1 and φ_2

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = A \begin{Bmatrix} 0.614 \\ 1.000 \end{Bmatrix} \cos(\omega_1 t + \varphi_1) + B \begin{Bmatrix} -1.618 \\ 1.000 \end{Bmatrix} \cos(\omega_2 t + \varphi_2)$$

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = -\omega_1 A \begin{Bmatrix} 0.614 \\ 1.000 \end{Bmatrix} \sin(\omega_1 t + \varphi_1) - \omega_2 B \begin{Bmatrix} -1.618 \\ 1.000 \end{Bmatrix} \sin(\omega_2 t + \varphi_2)$$

For $t = 0$

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = A \begin{Bmatrix} 0.614 \\ 1.000 \end{Bmatrix} \cos\varphi_1 + B \begin{Bmatrix} -1.618 \\ 1.000 \end{Bmatrix} \cos\varphi_2$$

5-18 Cont.

$$\begin{Bmatrix} 0 \\ V \end{Bmatrix} = -\omega_1 A \begin{Bmatrix} 0.614 \\ 1.000 \end{Bmatrix} \sin\varphi_1 - \omega_2 B \begin{Bmatrix} -1.618 \\ 1.000 \end{Bmatrix} \sin\varphi_2$$

From 1st two eqs. $\varphi_1 = \varphi_2 = 90°$

From 3rd eq. $\omega_1 A(0.614) = \omega_2 B(1.618)$

From 4th eq. $V = \frac{-1.618}{0.614}\omega_2 B - \omega_2 B = -3.635\omega_2 B$

$$\therefore B = \frac{-V}{3.635\omega_2} = -.2751\frac{V}{\omega_2}$$

$$A = \frac{1.618}{.614}\frac{\omega_2}{\omega_1}\left(-.2751\frac{V}{\omega_2}\right) = -0.7249\frac{V}{\omega_1}$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0.7249\frac{V}{\omega_1}\begin{Bmatrix} 0.614 \\ 1.000 \end{Bmatrix}\sin\omega_1 t + \frac{V}{3.635\omega_2}\begin{Bmatrix} -1.618 \\ 1.000 \end{Bmatrix}\sin\omega_2 t$$

where: $\omega_1 = 0.618\sqrt{\frac{k}{m}}$, $\omega_2 = 1.618\sqrt{\frac{k}{m}}$, $\varphi = \varphi_2 = 90°$

5-19 Using same gen. eqs. of Prob. 5-18

$$\begin{Bmatrix} 0 \\ 1.0 \end{Bmatrix} = A\begin{Bmatrix} 0.614 \\ 1.000 \end{Bmatrix}\cos\varphi_1 + B\begin{Bmatrix} -1.618 \\ 1.000 \end{Bmatrix}\cos\varphi_2$$

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = -\omega_1 A\begin{Bmatrix} 0.614 \\ 1.00 \end{Bmatrix}\sin\varphi_1 - \omega_2 B\begin{Bmatrix} -1.618 \\ 1.000 \end{Bmatrix}\sin\varphi_2$$

The second set of eqs. require $\varphi_1 = \varphi_2 = 0$

$\therefore$ $0.614A = 1.618B$ $\qquad B = .3795A$

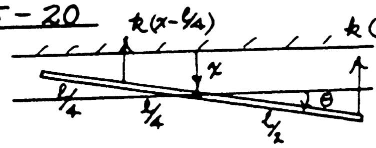
$1.0 = A + .3795A$ $\qquad \therefore A = .7249$

$B = .2751$

Subst. into gen. eqs.

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0.4451 \\ 0.7249 \end{Bmatrix}\cos\omega_1 t + \begin{Bmatrix} -0.4451 \\ 0.2751 \end{Bmatrix}\cos\omega_2 t$$

5-20



$$m\ddot{x} = -k(x+\tfrac{l}{2}\theta) - k(x-\tfrac{l}{4}\theta)$$

$$J\ddot{\theta} = -k(x+\tfrac{l}{2}\theta)\tfrac{l}{2} + k(x-\tfrac{l}{4}\theta)\tfrac{l}{4}$$

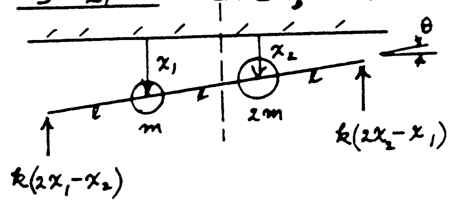
$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} 2k & k\frac{l}{4} \\ k\frac{l}{4} & k\frac{5l}{16} \end{bmatrix}\begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix}$$

$$\begin{vmatrix} (2-\frac{\omega^2 m}{k}) & \frac{l}{4} \\ \frac{1}{4} & (\frac{5}{16}-\frac{\omega^2 J}{kl}) \end{vmatrix} = 0$$

$$\omega^4\left(\frac{m}{k}\frac{J}{kl}\right) - \omega^2\left(\frac{5}{16}\frac{m}{k} + \frac{2J}{kl}\right) + \frac{9}{16} = 0$$

$$\frac{x}{\theta} = -4\left(\frac{5}{16} - \frac{\omega^2 J}{kl}\right)$$

5-21 $cm = \frac{l}{3}$ from $2m$



$$m\ddot{x}_1 + 2m\ddot{x}_2 = -k(2x_1-x_2) - k(2x_2-x_1)$$

$$J_{cm} = m\left(\tfrac{2}{3}l\right)^2 + 2m\left(\tfrac{l}{3}\right)^2 = \tfrac{2}{3}ml^2$$

$$\theta = \frac{x_1-x_2}{l}$$

Moment eq. $\frac{2}{3}ml^2\left(\frac{\ddot{x}_1-\ddot{x}_2}{l}\right) = k(2x_2-x_1)\frac{4l}{3} - k(2x_1-x_2)\frac{5l}{3}$

$$m\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + k\begin{bmatrix} 1 & 1 \\ 14 & -13 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix}$$

Let $\lambda = \frac{\omega^2 m}{k}$

$$\begin{vmatrix} (1-\lambda) & (1-2\lambda) \\ (14-2\lambda) & -(13-2\lambda) \end{vmatrix} = 0$$

$2\lambda^2 - 15\lambda + 9 = 0$

$\lambda_1 = 0.658 \qquad \omega_1 = 0.811\sqrt{\frac{k}{m}}$

$\lambda_2 = 6.842 \qquad \omega_2 = 2.62\sqrt{\frac{k}{m}}$

$$\frac{X_1}{X_2} = \frac{(1-2\lambda)}{-(1-\lambda)} \qquad \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_1 = \begin{Bmatrix} 0.921 \\ 1.00 \end{Bmatrix} \qquad \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_2 = \begin{Bmatrix} -2.17 \\ 1.00 \end{Bmatrix}$$

5-22 $2m(\ddot{x}_1 - l\ddot{\theta}) + m\ddot{x}_1 = -k(x_1+l\theta) - k(x_1-2l\theta)$

$$J_{cm}\ddot{\theta} = k(x_1-2l\theta)\frac{4l}{3} - k(x_1-l\theta)\frac{5l}{3}$$

Both static & dynamic coupling present.

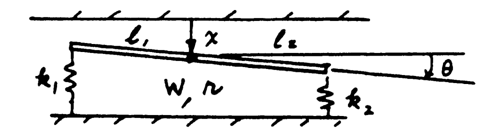
5-23 From Prob 5-9

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \frac{g}{\ell}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0 \quad \therefore \text{ Static coupling}$$

From Prob 5-10

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \frac{g}{\ell}\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = 0 \quad \therefore \text{ Static \& dynamic coupling}$$

5-24



Let $a = \frac{1}{m}(k_1 + k_2)$

$b = \frac{1}{m}(k_2\ell_2 - k_1\ell_1)$

$c = \frac{1}{mr^2}(k_1\ell_1^2 + k_2\ell_2^2)$

r = rad. of gyr. about cm.

$$\ddot{x} + ax + b\theta = 0$$

$$\ddot{\theta} + c\theta + \frac{b}{r^2}x = 0$$

$$\omega^4 - (a+c)\omega^2 + (ac - \frac{b^2}{r^2}) = 0$$

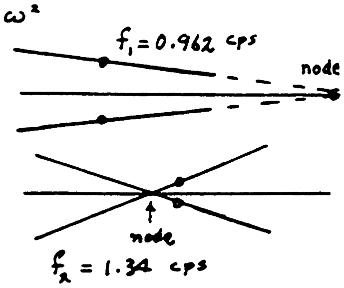
$$\frac{X}{\theta} = \frac{-b}{a - \omega^2}$$

For data given

$a = 40.5, \quad b = 42.8, \quad c = 65.6$

$\omega_1^2 = 36.57 \qquad \left(\frac{X}{\theta}\right)_1 = -10.9$ ft/rad

$\omega_2^2 = 69.53 \qquad \left(\frac{X}{\theta}\right)_2 = 1.47$ ft/rad.



5-25 Refering to Prob 5-24, $a = \omega_{11}^2$ is the uncoupled translational frequency squared and $c = \omega_{22}^2$ is the uncoupled rotational frequency squared. Thus the frequency equation becomes

$$\omega^4 - (\omega_{11}^2 + \omega_{22}^2)\omega^2 + (\omega_{11}^2\omega_{22}^2 - \frac{b^2}{r^2}) = 0$$

The solution is

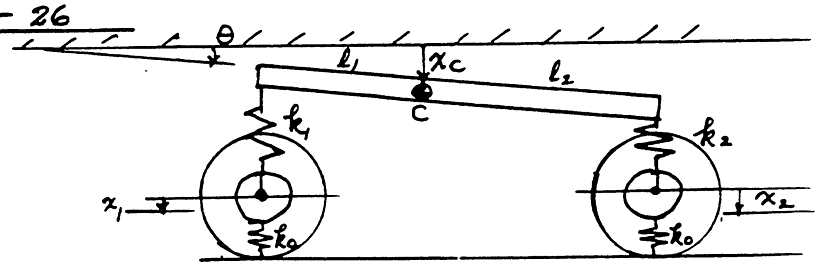
$$\omega^2 = \frac{1}{2}(\omega_{11}^2 + \omega_{22}^2) \pm \sqrt{\frac{1}{4}(\omega_{11}^2 - \omega_{22}^2)^2 + \frac{b^2}{r^2}}$$

But b is the coupling term, If it is zero we obtain the two uncoupled frequeneies. If $b \neq 0$ the radical is larger so that the uncoupled frequencies lie between the coupled frequencies

ie. with $b = 0 \qquad \omega^2 = \frac{1}{2}(\omega_{11}^2 + \omega_{22}^2) \pm \frac{1}{2}(\omega_{11}^2 - \omega_{22}^2) = \begin{cases} \omega_{11}^2 \\ \omega_{22}^2 \end{cases}$

with $b \neq 0 \qquad \omega^2 = \begin{cases} \omega_{11}^2 + \epsilon \\ \omega_{22}^2 - \epsilon \end{cases}$

5 - 26



Eqs. of motion

$$m\ddot{x}_c + k_1(x_c - l_1\theta - x_1) + k_2(x_c + l_2\theta - x_2) = 0$$

$$J_c\ddot{\theta} - k_1(x_c - l_1\theta - x_1)l_1 + k_2(x_c + l_2\theta - x_2)l_2 = 0$$

$$m_0\ddot{x}_1 - k_1[x_1 - (x_c - l_1\theta)] + k_0 x_1 = 0$$

$$m_0\ddot{x}_2 - k_2[x_c + l_2\theta - x_2] - k_0 x_2 = 0$$

These equations can be rearranged to the matrix form given.

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5 - 27

The natural frequency of the wheel-tire system is

$$\omega_n = \sqrt{\frac{k_0}{m_0}} = \sqrt{\frac{22000 \times 32.2}{80}} = 94.10 \text{ rad/s.}$$ which is

several times that of the body-spring system.

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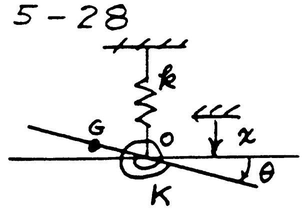
5 - 28



$$m(\ddot{x} - e\ddot{\theta}) + kx = 0$$

$$\Sigma M_G = J_G\ddot{\theta} + K\theta + kex = 0$$

$$J_0 = J_G + me^2$$

$$(J_0 - me^2)\ddot{\theta} + K\theta + kex = 0$$

---

5 - 29

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (k_1+k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix}\sin\omega t$$

$$\begin{bmatrix} 0.01 & 0 \\ 0 & 0.005 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 30 & -10 \\ -10 & 10 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix}\sin\omega t$$

5-29 Cont.

$$\begin{bmatrix} (30-0.01\omega^2) & -10 \\ -10 & (10-0.005\omega^2) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix}$$

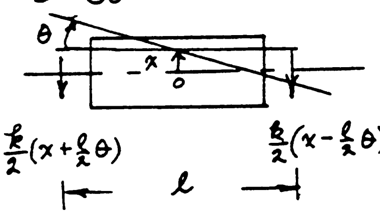
$$\omega^4 - 5000\,\omega^2 + 4\times 10^6 = 0$$

$$\omega^2 = 2500 \pm \sqrt{2500^2 - 4\times 10^6} = 2500 \pm \sqrt{2.25\times 10^6}$$

$$\omega_1 = 31.6 \text{ rad/s} \qquad \left(\frac{X_1}{X_2}\right)_1 = \frac{1}{2}$$

$$\omega_2 = 63.3 \text{ rad/s} \qquad \left(\frac{X_1}{X_2}\right)_2 = -1.0$$

5-30

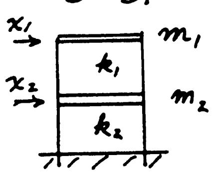


$$\Sigma F = -kx + mr\omega^2\cos\omega t = (M+m)\ddot{x} \cong M\ddot{x}$$

$$\Sigma M_0 = -\frac{k}{4}l^2\theta + mr\omega^2 b\cos\omega t = J_0\ddot{\theta}$$

$$m << M$$

5-31



$$m_1\ddot{x}_1 + k_1(x_1 - x_2) = 0$$

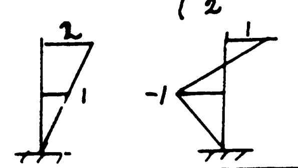
$$m_2\ddot{x}_2 - k_1(x_1 - x_2) + k_2 x_2 = 0$$

$$\text{Let } \lambda = \frac{\omega^2 m_1}{k_1}, \quad m_2 = 2m_1, \quad k_2 = 2k_1$$

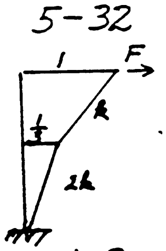
$$\left.\begin{aligned} (1-\lambda)x_1 - x_2 &= 0 \\ -x_1 + (3-2\lambda)x_2 &= 0 \end{aligned}\right\} \qquad \lambda^2 - \frac{5}{2}\lambda + 1 = 0$$

$$\lambda = \begin{cases} \frac{1}{2} \\ 2 \end{cases} \qquad \omega_1 = \sqrt{\frac{k_1}{2m_1}} \qquad \omega_2 = \sqrt{\frac{2k_1}{m_1}}$$

$$\frac{X_1}{X_2} = \frac{1}{1-\lambda} = \begin{Bmatrix} 2 \\ -1 \end{Bmatrix}$$



5-32



Initial Displ.

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 1 \end{Bmatrix} A\cos\omega_1 t + \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} B\cos\omega_2 t$$

Satisfies $\dot{x}_1(0) = \dot{x}_2(0) = 0$

$$\begin{Bmatrix} 1 \\ \frac{1}{3} \end{Bmatrix} = \begin{Bmatrix} 2 \\ 1 \end{Bmatrix} A + \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} B \qquad \therefore A = \frac{4}{9}, \quad B = -\frac{1}{9}$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 8/9 \\ 4/9 \end{Bmatrix}\cos\omega_1 t + \begin{Bmatrix} 1/9 \\ -1/9 \end{Bmatrix}\cos\omega_2 t.$$

5-33 From Prob 5-32

$x_1 = \frac{8}{9}\cos\omega_1 t + \frac{1}{9}\cos\omega_2 t \qquad \omega_1 = \sqrt{\frac{k_1}{2m_1}}$

$x_2 = \frac{4}{9}\cos\omega_1 t - \frac{1}{9}\cos\omega_2 t \qquad \omega_2 = \sqrt{\frac{2k_1}{m_1}} = 2\omega_1$

$$\left.\begin{array}{l} \text{shear in } 1^{st} \text{ story} = k_2 x_2 = 2k_1 x_2 \\ \quad " \quad " \quad 2^{nd} \quad " \quad = k_1(x_1 - x_2) \end{array}\right\} \text{Ratio} = \frac{2x_{2\,max}}{(x_1 - x_2)_{max}}$$

$$\frac{\partial x_2}{\partial t} = -\frac{4}{9}\omega_1 \sin\omega_1 t + \frac{1}{9}\omega_2 \sin\omega_2 t = 0$$

$$= -\frac{\omega_1}{9}\left\{4\sin\omega_1 t - 2\sin 2\omega_1 t\right\} = 0$$

$$= \left\{2\sin\omega_1 t - 2\sin\omega_1 t\cos\omega_1 t\right\} = 0$$

$\therefore \cos\omega_1 t = 1, \quad$ and $\omega_1 t = 0,\ 360^\circ \quad \therefore x_{2\,max} = \frac{4}{9} - \frac{1}{9} = \frac{1}{3}$

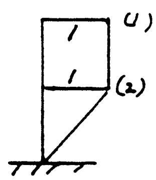
$$\frac{\partial(x_1 - x_2)}{\partial t} = -\frac{4}{9}\omega_1 \sin\omega_1 t - \frac{4}{9}\omega_1 \sin 2\omega_1 t = 0$$

or $\sin\omega_1 t(1 + 2\cos\omega_1 t) = 0, \quad \cos\omega_1 t = -\frac{1}{2}, \quad \omega_1 t = 120^\circ$

$$\therefore (x_1 - x_2)_{max} = \frac{4}{9}\left(-\frac{1}{2}\right) + \frac{2}{9}\left(-\frac{1}{2}\right) = \frac{1}{3}$$

$$\text{Ratio of shears} = \frac{2(\frac{1}{3})}{(\frac{1}{3})} = 2$$

5-34 See Prob. 5-32

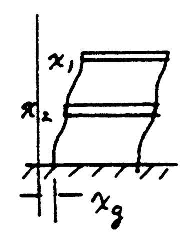


$$\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 1 \end{Bmatrix} A + \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} B$$

$$A = \frac{2}{3}, \qquad B = \frac{1}{3}$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 4/3 \\ 2/3 \end{Bmatrix}\cos\omega_1 t + \begin{Bmatrix} -1/3 \\ 1/3 \end{Bmatrix}\cos\omega_2 t$$

5-35 see Prob. 5-31 with $\lambda = \frac{\omega^2 m_1}{k_1}$



$$\begin{bmatrix} (1-\lambda) & -1 \\ -1 & (3-2\lambda) \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 2X_g \end{Bmatrix}$$

5-33 From Prob 5-32

$$x_1 = \frac{8}{9}\cos\omega_1 t + \frac{1}{9}\cos\omega_2 t \qquad \omega_1 = \sqrt{\frac{k_1}{2m_1}}$$

$$x_2 = \frac{4}{9}\cos\omega_1 t - \frac{1}{9}\cos\omega_2 t \qquad \omega_2 = \sqrt{\frac{2k_1}{m_1}} = 2\omega_1$$

$$\left.\begin{array}{l} \text{shear in } 1^{st} \text{ story} = k_2 x_2 = 2k_1 x_2 \\ \quad '' \quad '' \quad 2^{nd} \quad '' \quad = k_1(x_1 - x_2) \end{array}\right\} \text{Ratio} = \frac{2x_{2\,max}}{(x_1 - x_2)_{max}}$$

$$\frac{\partial x_2}{\partial t} = -\frac{4}{9}\omega_1 \sin\omega_1 t + \frac{1}{9}\omega_2 \sin\omega_2 t = 0$$

$$= -\frac{\omega_1}{9}\left\{4\sin\omega_1 t - 2\sin 2\omega_1 t\right\} = 0$$

$$= \left\{2\sin\omega_1 t - 2\sin\omega_1 t\cos\omega_1 t\right\} = 0$$

$$\therefore \cos\omega_1 t = 1, \quad \text{and} \quad \omega_1 t = 0,\ 360^\circ \quad \therefore x_{2\,max} = \frac{4}{9} - \frac{1}{9} = \frac{1}{3}$$

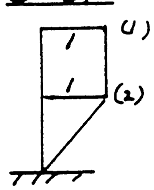
$$\frac{\partial(x_1 - x_2)}{\partial t} = -\frac{4}{9}\omega_1 \sin\omega_1 t - \frac{4}{9}\omega_1 \sin 2\omega_1 t = 0$$

$$\text{or} \quad \sin\omega_1 t(1 + 2\cos\omega_1 t) = 0, \quad \cos\omega_1 t = -\frac{1}{2}, \quad \omega_1 t = 120^\circ$$

$$\therefore (x_1 - x_2)_{max} = \frac{4}{9}\left(-\frac{1}{2}\right) + \frac{2}{9}\left(-\frac{1}{2}\right) = \frac{1}{3}$$

$$\text{Ratio of shears} = \frac{2(\frac{1}{3})}{(\frac{1}{3})} = 2$$

5-34 See Prob. 5-32

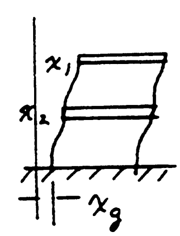


$$\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 1 \end{Bmatrix} A + \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} B$$

$$A = \frac{2}{3}, \qquad B = \frac{1}{3}$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 4/3 \\ 2/3 \end{Bmatrix}\cos\omega_1 t + \begin{Bmatrix} -1/3 \\ 1/3 \end{Bmatrix}\cos\omega_2 t$$

5-35 see Prob. 5-31 with $\lambda = \frac{\omega^2 m_1}{k_1}$



$$\begin{bmatrix} (1-\lambda) & -1 \\ -1 & (3-2\lambda) \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 2X_g \end{Bmatrix}$$

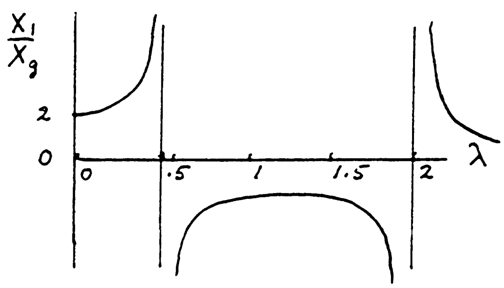
5-35 Cont. $\lambda_1 = \frac{1}{2} \qquad \lambda_2 = 2$

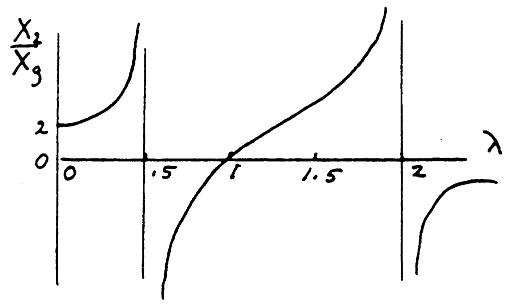
$\therefore$ Denominator $= (\lambda - \frac{1}{2})(\lambda - 2)$ use Cramer's rule

$$X_1 = \frac{\begin{vmatrix} 0 & -1 \\ 2 & (3-2\lambda) \end{vmatrix} X_g}{(\lambda-\frac{1}{2})(\lambda-2)} \qquad X_2 = \frac{\begin{vmatrix} (1-\lambda) & 0 \\ -1 & 2 \end{vmatrix} X_g}{(\lambda-\frac{1}{2})(\lambda-2)}$$

Response $= x_1 = X_1 \sin\omega t \qquad x_2 = X_2 \sin\omega t$

$$\frac{x_1}{X_g} = \frac{2\sin\omega t}{(\lambda-\frac{1}{2})(\lambda-2)} \qquad \frac{x_2}{X_g} = \frac{2(1-\lambda)\sin\omega t}{(\lambda-\frac{1}{2})(\lambda-2)}$$





5-36 $M(\ddot{y}_o - l_o\ddot{\theta}) = K_h(y_G - y_o)$

$$M\rho_c^2\ddot{\theta} = K_h(y_G - y_o)l_o - K_R\theta + Mgl_o\theta$$

ρ_c = rad. of gyration of bldg about its c.m.

5-37 Let $\omega_h^2 = \frac{K_h}{M} \qquad \omega_r^2 = \frac{K_r}{M\rho_c^2}$

$$\lambda = \frac{\omega}{\omega_h}, \quad \left(\frac{\rho_c}{l_o}\right)^2 = \frac{1}{3}, \quad \left(\frac{\omega_r}{\omega_h}\right)^2 = 4$$

$$\begin{bmatrix} (1-\lambda^2) & \lambda^2 \\ 1 & \frac{1}{3}(4-\lambda^2) \end{bmatrix} \begin{Bmatrix} Y_o \\ l_o\theta_o \end{Bmatrix} = Y_G e^{i\phi} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Characteristic eq.

$$\lambda^4 - 8\lambda^2 + 4 = 0$$

$$\lambda^2 = 4 \pm \sqrt{12} = \begin{cases} 0.54 \\ 7.46 \end{cases}$$

$$\lambda = \frac{\omega}{\omega_h} = \begin{cases} 0.734 \\ 2.73 \end{cases}$$

Ampl. ratio

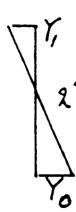
1st mode $\frac{Y_o}{l_o\theta_o} = \frac{\lambda^2 - 4}{3} = -1.15$

Let $Y_1 = Y_o - 2l_o\theta_o$ = displ of top, then $\frac{Y_1}{l_o\theta_o} = \frac{Y_o}{l_o\theta_o} - 2 = -1.15 - 2 = -3.15$

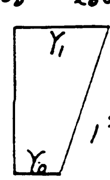
$$\frac{Y_1}{Y_o} = \frac{-3.15}{-1.15} = 2.73$$

2nd mode

$$\left.\begin{aligned} \frac{Y_o}{l_o\theta_o} &= 1.15 \\ \frac{Y_1}{l_o\theta_o} &= -.85 \end{aligned}\right\} \frac{Y_1}{Y_o} = -.74$$



2nd mode



1st mode

5-38 From Prob. 5-37 $Det. = \lambda^4 - 8\lambda^2 + 4$

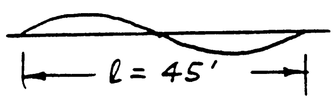
For $Det = 0$, $\lambda = \dfrac{\omega}{\omega_h} = \begin{cases} 0.732 & 1^{st}\ mode \\ 2.732 & 2^{nd}\ mode \end{cases}$

Using Cramer's rule

$$\left.\begin{aligned} \frac{Y_0}{Y_G} &= \frac{\begin{vmatrix} 1 & \lambda^2 \\ 1 & \frac{1}{3}(4-\lambda^2) \end{vmatrix}}{\lambda^4 - 8\lambda^2 + 4} = \frac{4}{3}\,\frac{(1-\lambda^2)}{\lambda^4 - 8\lambda^2 + 4} \\ \frac{l_0\theta_0}{Y_G} &= \frac{\begin{vmatrix} (1-\lambda^2) & 1 \\ 1 & 1 \end{vmatrix}}{\lambda^4 - 8\lambda^2 + 4} = \frac{-\lambda^2}{\lambda^4 - 8\lambda^2 + 4} \end{aligned}\right\}$$

Plot for various values of λ to Check Fig P5-38

5-39



Unfavorable speed is found from $v\tau = l$

$$v = \frac{l}{\tau} = l f_n$$

From Prob 5-24

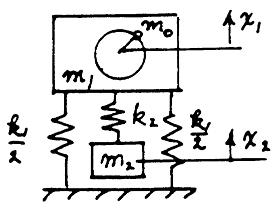
$f_1 = 0.962$ predominately bouncing

$f_2 = 1.327$ " pitching

$$v_1 = 45 \times 0.962 = 43.29\ ft/s = 29.5\ mph$$

$$v_2 = 45 \times 1.327 = 59.72\ ft/s = 40.7\ mph$$

5-40



$$m_1\ddot{x}_1 + k_1 x_1 + k_2(x_1 - x_2) = m_0 e\omega^2 \sin\omega t$$

$$m_2\ddot{x}_2 + k_2(x_2 - x_1) = 0$$

$\dfrac{k_2}{m_2}$ should equal exciting freq ω^2

$$k_2 = \omega^2 m_2 = \left(\frac{2\pi\,1800}{60}\right)^2\left(\frac{50}{386}\right) = 4600\ lb/in$$

The absorber will then make $x_1 = 0$ and the absorber force will be equal and opposite to the exciting force.

$$k_2 X_2 = m_0 e\omega^2$$

$$X_2 = \frac{m_0 e\,\omega^2}{k_2} = \frac{m_0 e}{m_2} = \frac{2}{50} = 0.04''$$

5-41

$$m_1\ddot{x}_1 + c(\dot{x}_1-\dot{x}_2) + k_1x_1 + k_2(x_1-x_2) = m_0e\omega^2(e^{i\omega t})$$

$$m_2\ddot{x}_2 + c(\dot{x}_2-\dot{x}_1) + k_2(x_2-x_1) = 0$$

Let $x_1 = X_1e^{i\omega t}$, $x_2 = X_2e^{i\omega t}$ where X_1 & X_2 are complex

$$\left(\frac{k_1+k_2}{m_1} - \omega^2 + i\frac{\omega c}{m_1}\right)X_1 - \left(\frac{k_2}{m_1} + \frac{i\omega c}{m_1}\right)X_2 = \frac{(m_0e)\omega^2}{m_1}$$

$$-\left(\frac{k_2}{m_2} + \frac{i\omega c}{m_2}\right)X_1 + \left(\frac{k_2}{m_2} - \omega^2 + \frac{i\omega c}{m_2}\right)X_2 = 0$$

$$X_1 = \frac{(m_0e)\omega^2(k_2 - m_2\omega^2 + i\omega c)}{[(k_1+k_2) - m_1\omega^2 + i\omega c][k_2 - m_2\omega^2 + i\omega c] - (k_2 + i\omega c)^2}$$

$$\frac{X_2}{X_1} = \frac{(k_2 - i\omega c)}{(k_1 - m_2\omega^2 + i\omega c)}$$

5-42

$$I\ddot{\theta}_1 + 4ka^2(\theta_1-\theta_2) = M_0\sin\omega t$$

$$I_d\ddot{\theta}_2 + 4ka^2(\theta_2-\theta_1) = 0$$

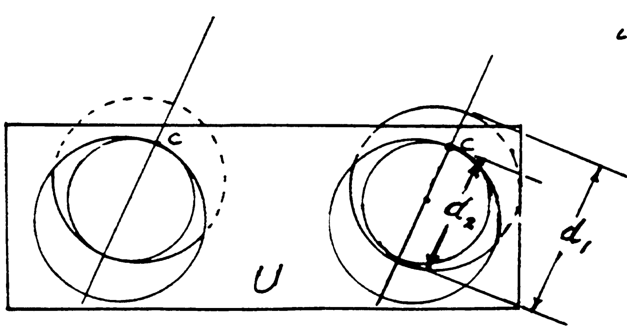
Let $\lambda = \frac{\omega^2 I}{ka^2}$ and $n = \frac{I_d}{I}$

$$\begin{bmatrix} (4-\lambda) & -1 \\ -1 & (4-n\lambda) \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} M_0 \\ 0 \end{Bmatrix}\sin\omega t$$

$$\theta_1 = \frac{(4-n\lambda)M_0\sin\omega t}{n\lambda^2 - 4(1+n)\lambda + 15} \qquad \therefore\ \theta_1 = 0 \text{ when } \lambda = \frac{4}{n}$$

5-43 Point of contact c of U with small circle moves in circle of radius

$r = d_1 - d_2$



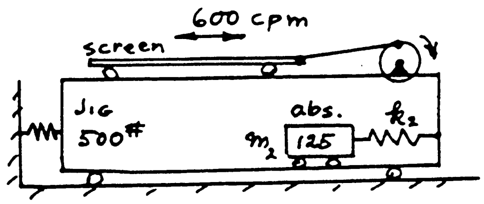
5-44 From Eq. 5.7-5 the natural freq. is

$$\omega_n = n\sqrt{\frac{R}{r}} = 4\,(\text{rot. speed}) = 4n$$

$$\therefore \frac{r}{R} = \frac{1}{16} \qquad r = d_1 - d_2 = \frac{3}{4}'' - d_2$$

$$\therefore r = \frac{R}{16} = \frac{4''}{16} = \frac{1}{4} = \frac{3}{4} - d_2 \qquad d_2 = \frac{1}{2}''$$

5-45



$f_1 = 400$ cpm.

Excit. $\omega = \frac{2\pi 600}{60} = 20\pi$ rad/s.

Nat. freq. of absorber must equal the excitation freq.

$$\omega_{22}^2 = \frac{k_2}{m_2} = \frac{386\,k_2}{125} = (20\pi)^2$$

$$\therefore k_2 = 1278. \ \#/\text{in}$$

Nat. freq. of system is found from the denominator of Eq 5.6-1 which can be reduced to

$$\left(\frac{\omega}{\omega_{22}}\right)^4 - \left[1 + \left(\frac{\omega_{11}}{\omega_{22}}\right)^2\left\{1 + \mu\left(\frac{\omega_{22}}{\omega_{11}}\right)^2\right\}\right]\left(\frac{\omega}{\omega_{22}}\right)^2 + \left(\frac{\omega_{11}}{\omega_{22}}\right)^2 = 0$$

$$\mu = \frac{m_2}{m_1} = \frac{125}{500} = 0.25, \qquad \left(\frac{\omega_{11}}{\omega_{22}}\right)^2 = \left(\frac{400}{600}\right)^2 = \frac{1}{2.25}, \qquad \text{Let } \lambda = \frac{\omega}{\omega_{22}}$$

$$\lambda^4 - 1.695\,\lambda^2 + \frac{1}{2.25} = 0, \qquad \lambda^2 = \left(\frac{\omega}{\omega_{22}}\right)^2 = 0.845 \pm .164$$

5-46 Refer to Fig 5.6-3

With trial weight of 2 lb tuned to 232 rpm, the two nat. freqs. are

$$\left(\frac{\omega}{\omega_{22}}\right) = \frac{198}{232} = 0.854 \quad \text{and} \quad \left(\frac{\omega}{\omega_{22}}\right) = \frac{272}{232} = 1.17$$

These numbers establish the mass ratio from Fig 5.6-3 to be $\mu \cong 0.10$

To move nat. freqs. outside specified freqs. of

$$\left(\frac{\omega}{\omega_{22}}\right) = \frac{160}{232} = 0.69 \quad \text{and} \quad \left(\frac{\omega}{\omega_{22}}\right) = \frac{320}{232} = 1.38$$

Fig 5.6-3 shows $\mu \geq 0.57$

5-46 Cont.

Since $\mu_1 = \frac{(m_2)_1}{m_1} = \frac{2}{m_1} = 0.10 \qquad \therefore m_1 = 20$

$$\mu_2 = \frac{(m_2)_2}{m_1} = 0.57 \,, \quad (m_2)_2 = .57 \times 20 = 11.4 \text{ lb.}$$

The stiffness should be $k_2 = m_2\omega^2 = \frac{11.4}{386}\left(\frac{2\pi\, 232}{60}\right)^2 = 17.9 \ \#/in$

5-47 Assume linear velocity distribution of fluid between disk and case. The torque is

$$T = \mu(\text{velocity gradient})(\text{radius})(\text{area})$$

$$= 2\int_{R_0}^{R} 2\pi\mu\left(\frac{\omega r}{h}\right) r^2 dr + 2\pi\mu\left(\frac{\omega R}{h}\right) R^2 b$$

$$= 2\pi \frac{\mu\omega R^3}{h}\left[\frac{1}{2}\left(R - \frac{R_0^4}{R^3}\right) + b\right]$$

5-48 Optimum damping given by Eq. 5.8-6

$$\zeta_0 = \frac{\mu}{\sqrt{2(1+\mu)(2+\mu)}} = \frac{.25}{\sqrt{2(1.25)(2.25)}} = 0.1054$$

The freq. at which the damper is most effective (with peak ampl.) is given by Eq. 5.8-7

$$\frac{\omega}{\omega_m} = \sqrt{\frac{2}{2+\mu}} = \sqrt{\frac{2}{2.25}} = 0.943$$

5-49 The peak amplitude for any μ and ζ can be found from Fig 5.8-4. It is seen that the optimum (lowest point on curve) for $\mu = .25$ is $\zeta \cong .105$ as computed in Prob. 5-48. Thus

$$\frac{\theta_{max\ \zeta=.10}}{\theta_{max\ \zeta=.105}} \cong 1.0$$

5-50 In Fig 5.8-3 all curves pass through a common point p. This point is at the frequency for optimum damping. Thus by equating $\left|\frac{K\theta_0}{M_0}\right|^2$ for $\zeta = 0$ and $\zeta = \infty$ Eq. 5.8-7 is obtained

$$\frac{\mu^2(\omega/\omega_m)^2}{\mu^2(\omega/\omega_m)^2(1-\omega^2/\omega_m^2)} = \frac{4}{4\left[\mu\left(\frac{\omega}{\omega_m}\right)^2-(1-\omega^2/\omega_m^2)\right]^2}$$

or. $$\frac{1}{1-(\omega/\omega_m)^2} = -\frac{1}{(\omega/\omega_m)^2(1+\mu)-1}$$

Thus $$\omega/\omega_m = \sqrt{\frac{2}{2+\mu}}$$

Eq. 5.8-6 is found by differentiating $\left|\frac{K\theta_0}{M_0}\right|^2$ w.r.t. $\left(\frac{\omega}{\omega_m}\right)^2$, equating it to zero and substituting Eq. 5.8-7

Let $r^2 = \frac{2}{2+\mu} = \left(\frac{\omega}{\omega_m}\right)^2$ $\quad (1-r^2) = \frac{\mu}{2+\mu} = r^2(1+\mu) - 1$

$q = \frac{4\zeta^2}{\mu^2}$ Rewrite Eq. 5.8-5

$$\left|\frac{K\theta_0}{M_0}\right|^2 = \frac{r^2+q}{r^2(1-r^2)^2+q\left[r^2(1+\mu)-1\right]^2}$$

$$\frac{\partial|\;|^2}{\partial r^2} = r^2(1-r^2)^2+q\left[r^2(1+\mu)-1\right]^2-(r^2+q)\Big\{(1-r^2)^2-2r^2(1-r^2)$$
$$+2q\left[r^2(1+\mu)-1\right](1+\mu)\Big\} = 0$$

$$= r^2(1-r^2)^2+q(1-r^2)^2-(r^2+q)\{\quad \text{same} \quad\} = 0$$

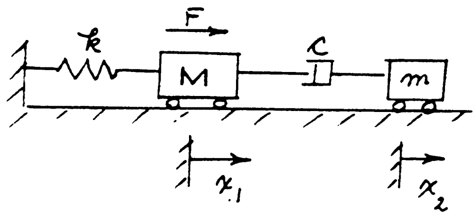
$$= (r^2+q)\left[\cancel{\left(\frac{\mu}{2+\mu}\right)^2}-\cancel{\left(\frac{\mu}{2+\mu}\right)^2}+2r^2\left(\frac{\mu}{2+\mu}\right)-2q\left(\frac{\mu}{2+\mu}\right)(1+\mu)\right] = 0$$

$$\therefore\; r^2-q(1+\mu)=0 \quad \text{and} \quad q = \frac{r^2}{1+\mu} = \frac{4\zeta^2}{\mu^2}$$

$$\zeta^2 = \frac{\mu^2}{4}\,\frac{1}{1+\mu}\,\frac{2}{2+\mu} = \frac{\mu^2}{2(1+\mu)(2+\mu)}.$$

$$\zeta_{opt} = \frac{\mu}{\sqrt{2(1+\mu)(2+\mu)}}$$

5-51



$$M\ddot{x}_1 = -kx_1 - c(\dot{x}_1 - \dot{x}_2) + Fe^{i\omega t}$$

$$m\ddot{x}_2 = c(\dot{x}_1 - \dot{x}_2) \qquad \text{Let } x_1 = X_1 e^{i\omega t}, \quad x_2 = X_2 e^{i\omega t}$$

$$\left[\left(\frac{k}{M} - \omega^2\right) + i\left(\frac{c\omega}{M}\right)\right]X_1 - i\left[\frac{c\omega}{M}\right]X_2 = \frac{F}{M}$$

$$\left[-\omega^2 + i\frac{c\omega}{m}\right]X_2 = i\left(\frac{c\omega}{m}\right)X_1 \qquad \text{eliminate } X_2$$

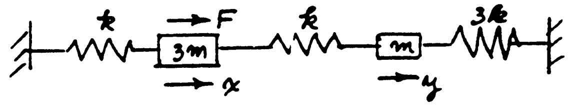
$$\left\{\left[\left(\frac{k}{M} - \omega^2\right) + i\left(\frac{c\omega}{M}\right)\right] - i\left(\frac{c\omega}{M}\right)\frac{i\left(\frac{c\omega}{m}\right)}{\left(-\omega^2 + \frac{ic\omega}{m}\right)}\right\}X_1 = \frac{F}{M}$$

$$\left\{\left[(k - M\omega^2) + ic\omega\right]\left(\frac{-m\omega^2 + ic\omega}{\cancel{m}}\right) + \frac{(c\omega)^2}{\cancel{m}}\right\}X_1 =$$

$$F\left(\frac{-m\omega^2 + ic\omega}{\cancel{m}}\right)$$

$$\therefore \frac{X_1}{F} = \frac{(\omega^2 m - ic\omega)}{m\omega^2(k - M\omega^2) + ic\omega\left[m\omega^2 - (k - M\omega^2)\right]}$$

5-52



$$\left.\begin{aligned}\ddot{x} &= -\tfrac{2}{3}\tfrac{k}{m}x + \tfrac{1}{3}\tfrac{k}{m}y + \tfrac{F}{3m}\\ \ddot{y} &= \tfrac{k}{m}x - 4\tfrac{k}{m}y\end{aligned}\right\}\qquad \begin{aligned}\omega_1 &= 0.751\sqrt{\tfrac{k}{m}}\\ \omega_2 &= 2.04\sqrt{\tfrac{k}{m}}\end{aligned}$$

Let $k = m = 1$, then $\tau_1 = 8.35$ sec, $\tau_2 = 3.07$ sec.

Let $\Delta t = 0.20$

$$x_{i+1} = \ddot{x}_i\,\Delta t^2 + 2x_i - x_{i-1}$$
$$y_{i+1} = \ddot{y}_i\,\Delta t^2 + 2y_i - y_{i-1}$$

Initial Cond. ($I=1$)

$$x(1) = y(1) = \ddot{y}(1) = 0, \qquad \ddot{x}_1(1) = \frac{100}{3}$$

From Eq. (4.7-8) $\quad x(2) = \frac{1}{2}(0.20)^2\left(\frac{100}{3}\right) = 0.666$

From Eq (4.7-12) and DE for $\ddot{y}$

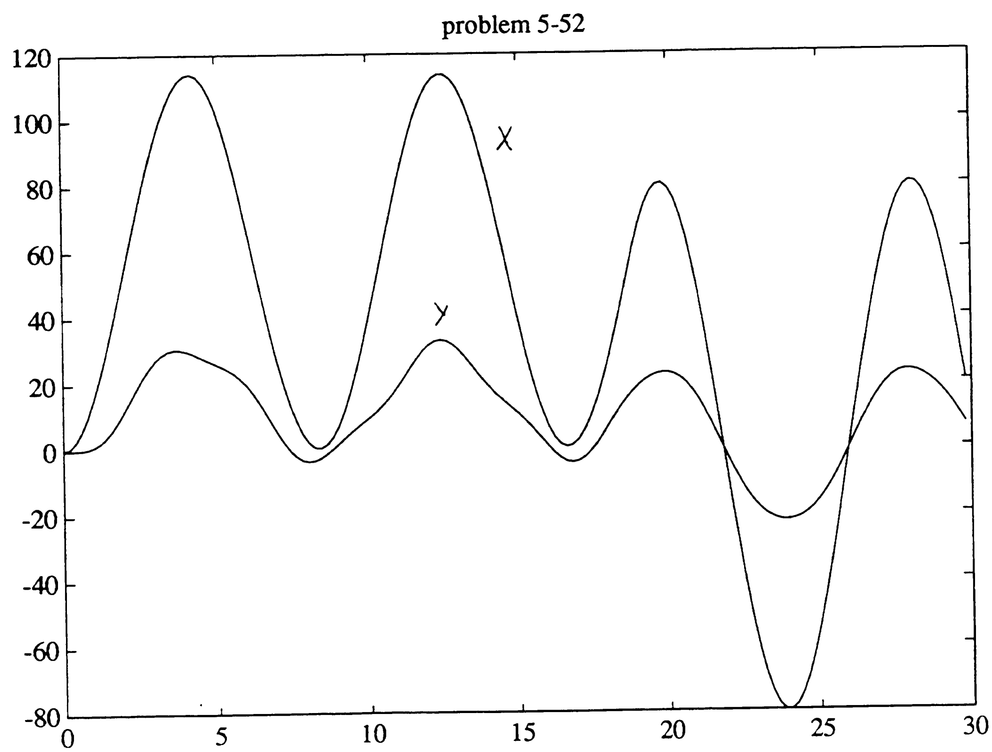
$$y(2) = \frac{1}{6}\Delta t^2\,\ddot{y}(2) = \frac{\Delta t^2}{6}\left[x(2) - 4y(2)\right]$$

$$y(2)\left[1 + \frac{2}{3}\Delta t^2\right] = \frac{1}{6}\Delta t^2 \times 0.666 \qquad \therefore\ y(2) = 0.00432$$

computer program and plot of results follow.

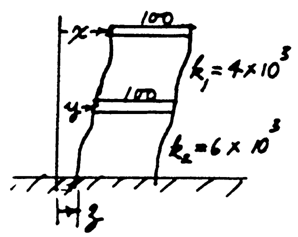
5-52 cont

```
%This is the program for problem 5-52
clear;clg;
dt=0.20;
f=100
x(1)=0;
y(1)=0;
dy(1)=0.0;
dx(1)=f/3;
t(1)=0;
x(2)=0.5*dx(1)*dt^2;
y(2)=(x(2)*dt^2)/(6+4*dt^2);
dx(2)=(-2*x(2)+y(2)+f)/3;
dy(2)=x(2)-4*y(2);
t(2)=dt
for i=3:150
if i<95
f=100;
else
f=0;
end
t(i)=(i-1)*dt;
x(i)=dx(i-1)*dt^2+2*x(i-1)-x(i-2);
y(i)=dy(i-1)*dt^2+2*y(i-1)-y(i-2);
dx(i)=(-2*x(i)+y(i)+f)/3;
dy(i)=x(i)-4*y(i);
end
```



5-53

$$100\ddot{x} = -4\times10^3(x-y)$$
$$100\ddot{y} = 4\times10^3(x-y) - 6\times10^3(y-z)$$



$$\therefore\ \ddot{x} = -40x + 40y$$
$$\ddot{y} = 40x - 100y + 60z \qquad z = 10\sin\pi t$$

Use Eq. 4.7-12 to start computation

$$x(2) = \frac{1}{6}\ddot{x}(2)h^2 = \frac{h^2}{6}[-40x(2)+40y(2)]$$

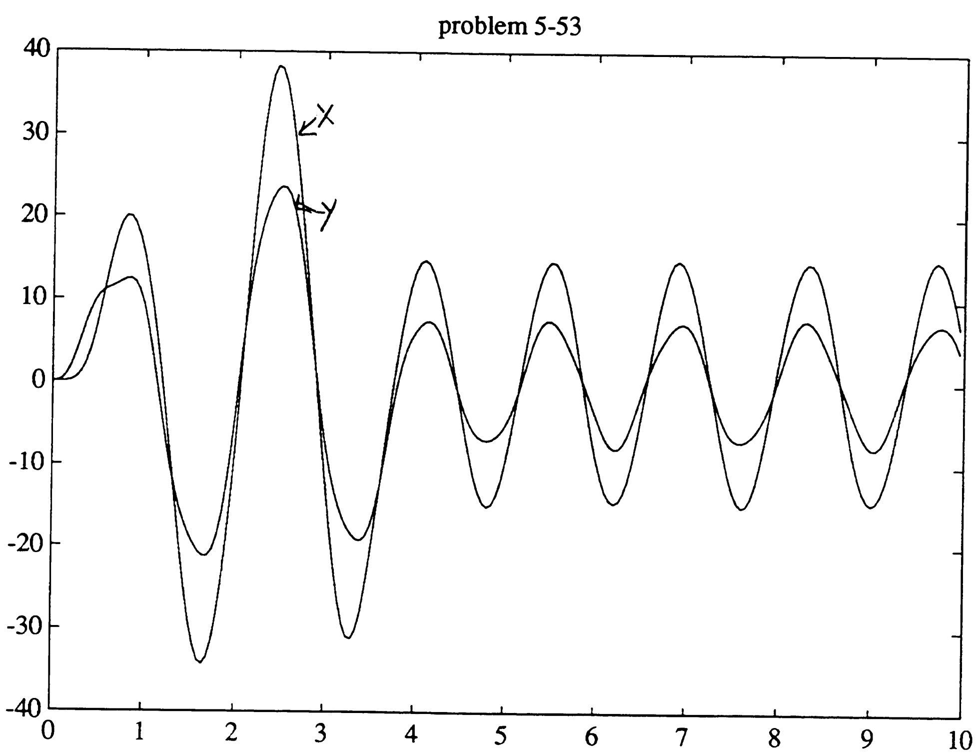
$$y(2) = \frac{h^2}{6}\ddot{y}(2) = \frac{h^2}{6}[40x(2)-100y(2)+60z(2)]$$

$$x(2)\left[1+\frac{40}{6}h^2\right] = y(2)\frac{40}{6}h^2$$

$$y(2)\left[1+\frac{100}{6}h^2\right] = x(2)\frac{40}{6}h^2 + 10h^2 z(2)$$

$$\therefore\ x(2) = \frac{40}{6}h^2\,\frac{1}{1+\frac{40}{6}h^2}\cdot\frac{1}{1+\frac{100}{6}h^2}\cdot\left[x(2)\frac{40}{6}h^2 + 10h^2 z(2)\right]$$

$$\left.\begin{aligned} x(2) &= 2400h^4 z(2)/(36+840h^2+2400h^4) \\ y(2) &= \left(1+\frac{40}{6}h^2\right)x(2)\Big/\frac{40}{6}h^2 \end{aligned}\right\}\ \text{starting equations}$$



5-53 cont.

```
%This is the matlab code for problem 5-53
clear;clg;
dt=0.05;
f=100;
x(1)=0.0;
y(1)=0.0;
dy(1)=0.0;
dx(1)=0.0;
t(1)=0;
z(1)=0;
t(2)=dt;
z(2)=10*sin(pi*dt);
x(2)=(400*6*dt^4*z(2))/(36+840*dt^2+2400*dt^4);
y(2)=(1+40/6*dt^2)*x(2)/(40/6*dt^2);
dx(2)=-40*x(2)+40*y(2);
dy(2)=40*x(2)-100*y(2)+60*z(2);
for i=3:200
t(i)=(i-1)*dt;
if t(i)<4
z(i)=10*sin(pi*t(i));
else
z(i)=0;
end
x(i)=dx(i-1)*dt^2+2*x(i-1)-x(i-2);
y(i)=dy(i-1)*dt^2+2*y(i-1)-y(i-2);
dx(i)=-40*x(i)+40*y(i);
dy(i)=40*x(i)-100*y(i)+60*z(i);
end
```

5-54

$$J_1 \ddot{\theta}_1 = K_1(\theta_2 - \theta_1)$$
$$J_2 \ddot{\theta}_2 = -K_1(\theta_2 - \theta_1) + K_2(\theta_3 - \theta_2)$$
$$J_3 \ddot{\theta}_3 = -K_2(\theta_3 - \theta_3)$$

If $\phi = (\theta_2 - \theta_1)$
$\psi = (\theta_3 - \theta_2)$
the DEs can be written as

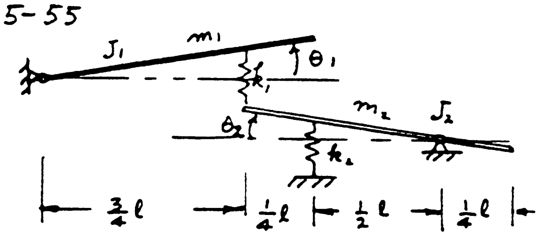
$$J_2 \ddot{\phi} = -k_1\left(1 + \frac{J_2}{J_1}\right)\phi + K_2 \psi$$
$$J_3 \ddot{\psi} = K_1 \frac{J_3}{J_2} \phi - K_2\left(1 + \frac{J_3}{J_2}\right)\psi$$

DEs in two coordinates

Freq. Eq. from either set of equations is

$$\omega^4 - \left[\frac{K_1}{J_1} + \frac{K_2}{J_2}\left(1 + \frac{K_1}{K_2} + \frac{J_2}{J_3}\right)\right]\omega^2 + \frac{K_1}{J_1}\frac{K_2}{J_2}\left(\frac{J_1 + J_2 + J_3}{J_3}\right) = 0$$

5-55



$$J_1 \ddot{\theta}_1 = -\tfrac{3}{4} l k_1 (\theta_1 - \theta_2)\tfrac{3}{4} l$$
$$J_2 \ddot{\theta}_2 = \tfrac{3}{4} l k_1 (\theta_1 - \theta_2)\tfrac{3}{4} l - \tfrac{1}{2} l k_2 \theta_2 \tfrac{1}{2} l$$

$$\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + l^2 k_1 \begin{bmatrix} \frac{9}{16} & -\frac{9}{16} \\ -\frac{9}{16} & \left(\frac{9}{16} + \frac{1}{4}\frac{k_2}{k_1}\right) \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix}$$

$$J_1 = m_1 l^2 \frac{1}{3} \qquad J_2 = m_2 l^2\left(\frac{1}{12} + \frac{1}{16}\right)$$

5-56

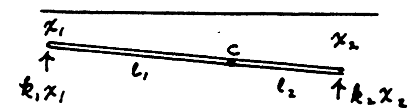
$$\begin{bmatrix} \left[K_1\left(1 + \frac{J_2}{J_1}\right) - \omega^2 J_2\right] & -K_2 \\ -K_1 \frac{J_3}{J_2} & \left[K_2\left(1 + \frac{J_3}{J_2}\right) - \omega^2 J_3\right] \end{bmatrix}\begin{Bmatrix} \phi \\ \psi \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix}$$

$$\frac{\phi}{\psi} = \frac{K_2}{K_1\left(1 + \frac{J_2}{J_1}\right) - \omega^2 J_2} = \frac{\theta_2 - \theta_1}{\theta_3 - \theta_2} = \frac{\frac{\theta_2}{\theta_1} - 1}{\frac{\theta_3}{\theta_1} - \frac{\theta_2}{\theta_1}}$$

<u>Assign numerical</u> values for K_s & J_s and prove

$$J_1 \theta_1^{(i)} \theta_1^{(j)} + J_2 \theta_2^{(i)} \theta_2^{(j)} + J_3 \theta_3^{(i)} \theta_3^{(j)} = 0$$

5-57



$$\text{Displ. of } c = x_1 + \frac{\ell_1}{\ell_1+\ell_2}(x_2 - x_1) = \frac{\ell_2 x_1 + \ell_1 x_2}{\ell_1 + \ell_2}$$

$$T = \frac{1}{2} m \left(\frac{\ell_2 \dot{x}_1 + \ell_1 \dot{x}_2}{\ell_1+\ell_2} \right)^2 + \frac{1}{2} J_c \left(\frac{\dot{x}_2 - \dot{x}_1}{\ell_1+\ell_2} \right)^2$$

$$U = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 \qquad \therefore \text{ Dynamic coupling exists}$$

5-58. Equations to be solved are:

$$\ddot{x} = -720\, x + 360\, y \qquad \text{(see Sec. 5.5)}$$
$$\ddot{y} = 1440(x-y) + 160$$

$$(s^2+720)\,\bar{x}(s) - 360\,\bar{y}(s) = 0$$
$$-1440\,\bar{x}(s) + (s^2+1440)\,\bar{y}(s) = \frac{160}{s}$$

from 1st eq. $\quad \bar{x}(s) = \left(\dfrac{360}{s^2+720}\right)\bar{y}(s)$

subst. into 2nd eq. $\quad -1440\left(\dfrac{360}{s^2+720}\right)\bar{y}(s) + (s^2+1440)\bar{y}(s) = \dfrac{160}{s}$

x by (s^2+720)

$$\left[-518{,}400 + s^4 + 2160 s^2 + 1{,}036{,}800\right]\bar{y}(s) = \frac{160(s^2+720)}{s}$$

$$\left[s^4 + 2160 s^2 + 518{,}400\right]\bar{y}(s) = \frac{160}{s}(s^2+720)$$

$$\bar{y}(s) = \frac{160(s^2+720)}{s(s^4+2160 s^2+518{,}400)}$$

Roots: $\quad s^2 = -1080 \pm \sqrt{1{,}166{,}400 - 518{,}400}$
$\quad = -1080 \pm \sqrt{648{,}000}$
$\quad = -1080 \pm 804.9845$

also $s = 0$ is a root.

5-58 Cont.

$$\therefore \quad s_1^2 = -275.0155 \qquad s_1 = \pm i\,16.5836$$

$$s_2^2 = -1884.9845 \qquad s_2 = \pm i\,43.4164$$

Rewrite $\bar{y}(s)$ as

$$\bar{y}(s) = \frac{160\,(s^2+720)}{s\,(s^2+275.0155)(s^2+1884.9845)}$$

$$= \frac{160\,(s^2+720)}{s(s+i\,16.5836)(s-i\,16.5836)(s+i\,43.4164)(s-i\,43.4164)}$$

$$= \frac{C_1}{(s+i\,16.5836)} + \frac{C_2}{(s-i\,16.5836)} + \frac{C_3}{(s+i\,43.4164)}$$

$$+ \frac{C_4}{(s-i\,43.4164)} + \frac{C_5}{s}$$

$$C_1 = \lim_{s\to s_{1-}} \left[\frac{160\,(s^2+720)\cancel{(s+i\,16.5836)}}{s\,\cancel{(s+i\,16.5836)}(s-i\,16.58)(s^2+1884.9845)}\right]$$

$$= \frac{160\,(-275.0155+720)}{(-i\,16.5836)(-i\,33.1672)(-275.0155+1884.9845)} = -.0804007$$

$$C_2 = \lim_{s\to s_{1-}} \left[\frac{160\,(s^2+720)\cancel{(s-i\,16.5836)}}{s(s+i\,16.5836)\cancel{(s-i\,16.5836)}(s^2+1884.98)}\right] = -.0804007$$

$$C_3 = \lim_{s\to s_{2-}} \left[\frac{160(s^2+720)\,(s+i\,43.4164)}{s\,(s^2+275.0155)(s+i\,43.4164)(s-i\,43.4164)}\right] = -.03071$$

$$C_4 = C_3$$

$$C_5 = \frac{160\,(720)}{(275.0155)(1884.9845)} = 0.2222$$

5-58 Cont.

$$\bar{y}(s) = \frac{.2222}{s} - \frac{.0804007}{(s+i\,16.5836)} - \frac{.0804007}{(s-i\,16.5836)}$$

$$- \frac{.03071}{(s+i\,43.4164)} - \frac{.03071}{(s-i\,43.4164)}$$

Inverse

$$y(t) = 0.2222 - .0804007\left(e^{-i16.5836} + e^{i16.5836}\right)$$

$$- .03071\left(e^{-i43.4164} + e^{i43.4164}\right)$$

$$= 0.2222 - .1608 \cos 16.58t - .06142 \cos 43.41t$$

$$= .1608(1 - \cos 16.58t) + .06142(1 - \cos 43.41t) \text{ meters}$$

move dec. pt. 2 places to left for centimeters

Solution for $x(t)$

$$\bar{x}(s) = \frac{360}{(s^2+720)}\,\bar{y}(s) = \frac{360}{\cancel{(s^2+720)}} \cdot \frac{160\cancel{(s^2+720)}}{s(s^2+275.0155)(s^2+1884.9845)}$$

∴ roots are same. Solve as above with partial fractions, the equation

$$\bar{x}(s) = \frac{57600}{s(s^2+275.0155)(s^2+1884.9845)}$$

Ans.

$$x(t) = .1301(1 - \cos 16.58t) - .0190(1 - \cos 43.41t) \text{ meters.}$$

5-59 Examine for example the subsidiary sol. for Prob 5-58

$$\bar{y}(s) = \sum \frac{c_i}{s - s_i} \quad \text{where } s_i \text{ are roots}$$

∴ Results are sums of normal modes

5-60

$$m\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{Bmatrix} + k\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\end{Bmatrix} = \begin{Bmatrix}F_0\\ 0\end{Bmatrix}\sin\omega t$$

$$ms^2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}\bar{x}_1(s)\\ \bar{x}_2(s)\end{Bmatrix} + k\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}\begin{Bmatrix}\bar{x}_1(s)\\ \bar{x}_2(s)\end{Bmatrix} = \begin{Bmatrix}\frac{F_0\omega}{s^2+\omega^2} + msx_1(0) + m\dot{x}_1(0)\\ msx_2(0) + m\dot{x}_2(0)\end{Bmatrix}$$

$$\begin{bmatrix}(ms^2+2k) & -k\\ -k & (ms^2+2k)\end{bmatrix}\begin{Bmatrix}\bar{x}_1(s)\\ \bar{x}_2(s)\end{Bmatrix} = \begin{Bmatrix} & '' & \end{Bmatrix}$$

Det. of matrix $(ms^2+2k)^2 - k^2 = 0$, $s^2 = -\frac{2k}{m} \pm \frac{k}{m}$

$$\therefore \left(s^2+\frac{k}{m}\right)\left(s^2+\frac{3k}{m}\right)m$$

$$\bar{x}_1(s) = \frac{\begin{vmatrix}\left[\frac{F_0\omega}{s^2+\omega^2} + msx_1(0) + m\dot{x}(0)\right] & -k\\ \left[msx_2(0) + m\dot{x}_2(0)\right] & (ms^2+2k)\end{vmatrix}}{\left(s^2+\frac{k}{m}\right)\left(s^2+\frac{3k}{m}\right)m}$$

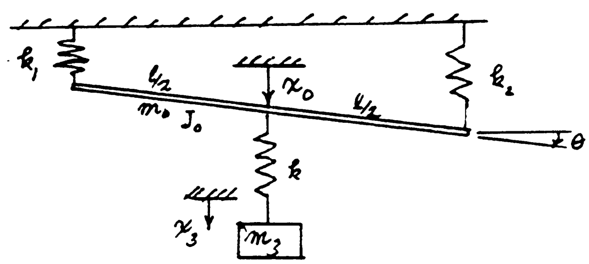
$$\bar{x}_2(s) = \frac{\begin{vmatrix}(ms^2+2k) & \left[\frac{F_0\omega}{s^2+\omega^2} + msx_1(0) + m\dot{x}_1(0)\right]\\ -k & \left[msx_2(0) + m\dot{x}_2(0)\right]\end{vmatrix}}{\left(s^2+\frac{k}{m}\right)\left(s^2+\frac{3k}{m}\right)m}$$

For steady state vibration $x_i(0)$, $\dot{x}(0) = 0$ and $s = i\omega$

$$x_1(t) = \frac{\left(2\frac{k}{m} - \omega^2\right)F_0\sin\omega t}{\left(\frac{k}{m} - \omega^2\right)\left(3\frac{k}{m} - \omega^2\right)}$$

$$x_2(t) = \frac{kF_0\sin\omega t}{\left(\frac{k}{m} - \omega^2\right)\left(3\frac{k}{m} - \omega^2\right)m}$$

5-61



$$m_0\ddot{x}_0 = -k_1(x_0 - \tfrac{\ell}{2}\theta) - k_2(x_0 + \tfrac{\ell}{2}\theta) + k_3(x_3 - x_0)$$

$$J_0\ddot{\theta} = k_1(x_0 - \tfrac{\ell}{2}\theta)\tfrac{\ell}{2} - k_2(x_0 + \tfrac{\ell}{2}\theta)\tfrac{\ell}{2}$$

$$m_3\ddot{x}_3 = -k_3(x_3 - x_0)$$

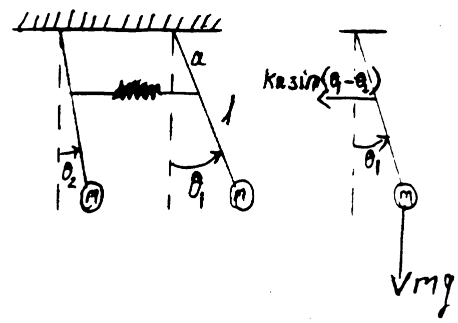
$$\begin{bmatrix} m_0 & & \\ & J_0 & \\ & & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_0 \\ \ddot{\theta} \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} (k_1+k_2+k_3) & (k_2-k_1)\frac{\ell}{2} & -k_3 \\ (k_2-k_1)\frac{\ell}{2} & (k_1+k_2)(\frac{\ell}{2})^2 & 0 \\ -k_3 & 0 & k_3 \end{bmatrix} \begin{Bmatrix} x_0 \\ \theta \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

5-62

$$\begin{bmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} (k_1+k_2+k_5) & -k_2 & -k_5 \\ -k_2 & (k_2+k_3+k_4) & -k_3 \\ -k_5 & -k_3 & (k_3+k_5) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

5-63

Not assuming small oscillations, we have

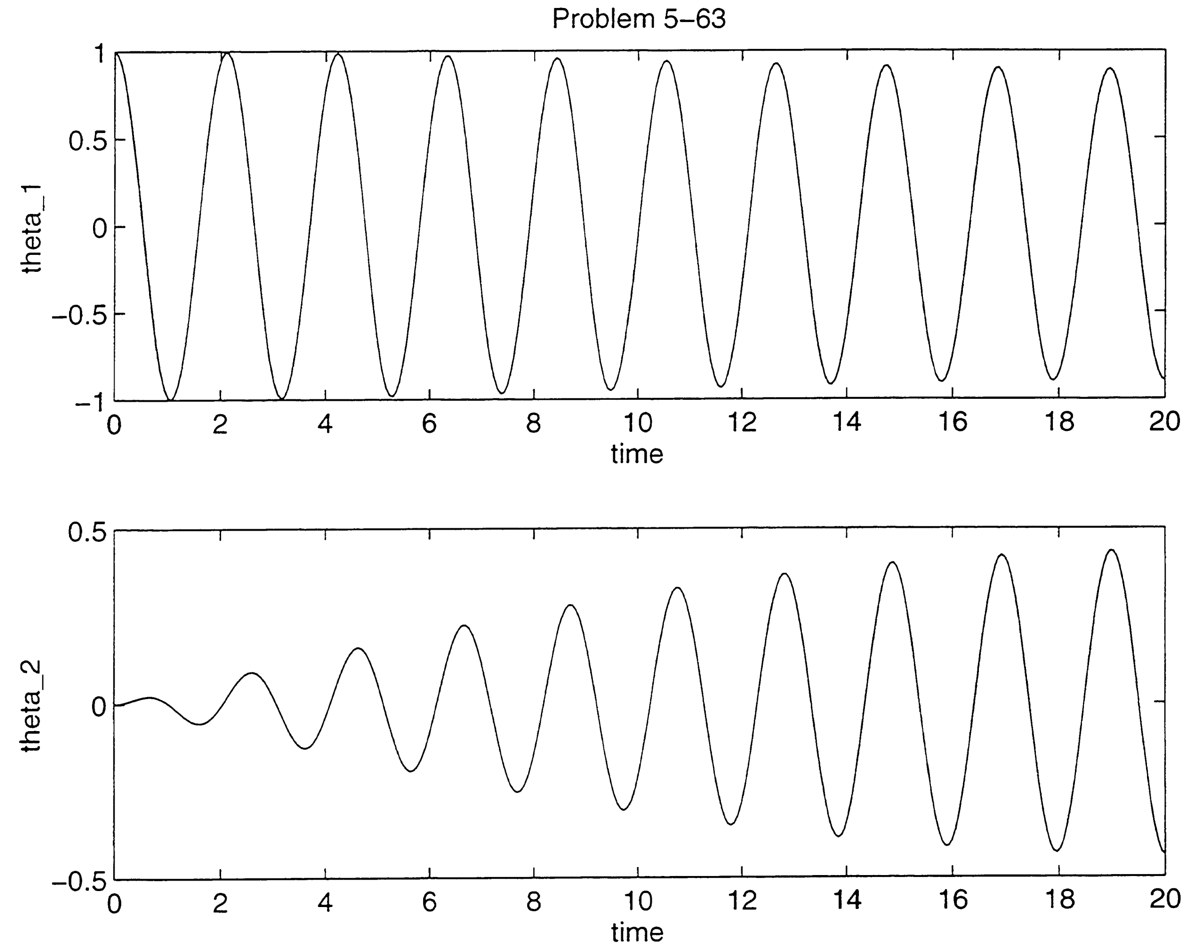


$$m l^2 \ddot{\theta}_1 = -mgl\sin\theta_1 - ka^2\sin(\theta_1-\theta_2)$$
$$m l^2 \ddot{\theta}_2 = -mgl\sin\theta_2 + ka^2\sin(\theta_1-\theta_2)$$

```
clear
global m l a k g
'Problem 5-63 Solution'
m=input('Enter the value of the mass (kg) : ')
k=input('Enter the value of the spring stiffness (N/m): ')
l=input('Enter the value of the pendulum length (m) : ')
a=input('Enter the value of the distance a (m): ')
x0=input('Enter the four dimensional initial condition [x1 x1d x2 x2d] : ')
tf=input('Enter the value of the final time (s) : ')
t0=0;g=9.81;
[t,x]=ode45('p563d',t0,tf,x0);
subplot(2,1,1);plot(t,x(:,1));xlabel('time');ylabel('theta_1');title('Problem 5-63')
subplot(2,1,2);plot(t,x(:,3));xlabel('time');ylabel('theta_2');
```

```
function xdot=p563d(t,x)
global m l a k g
xdot=[x(2);
        -(g/l)*sin(x(1))-(k/m)*(a/l)^2*sin(x(1)-x(3));
    x(4);
        -(g/l)*sin(x(3))+(k/m)*(a/l)^2*sin(x(1)-x(3))];
```

$m = k = l = 1$, $a = 0.5$, $X_0 = [1 \quad 0 \quad 0 \quad 0]$



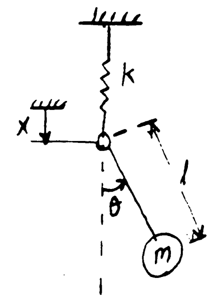
5-64

$f_i(x_1, x_2) = f_i(x_1 - x_1, x_2 - x_1) = f_i(0, x_2 - x_1)$, $i = 1, 2$

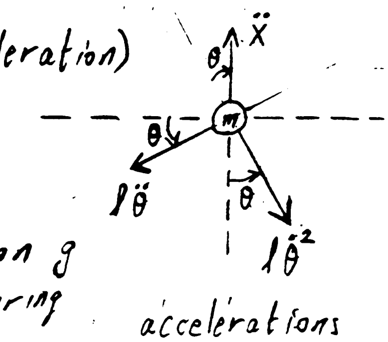
Let $y = x_2 - x_1$. Therefore

$$\ddot{y} = \ddot{x}_2 - \ddot{x}_1 = f_2(x_1, x_2) - f_1(x_1, x_2)$$
$$= f_2(0, x_2 - x_1) - f_1(0, x_2 - x_1)$$
$$= f_2(0, y) - f_1(0, y) = f(0, y).$$

5 - 65



Since the rod that connects the spring and the mass m is massless the spring force is transmitted from one end of the rod to the other. Thus, in the vertical direction, we have



$m a_v = -kx$ (a_v is the vertical acceleration)

$m(\ddot{x} - l\ddot{\theta}\sin\theta - l\dot{\theta}^2\cos\theta) = -kx$

$\therefore \quad \ddot{x} - l\ddot{\theta}\sin\theta - l\dot{\theta}^2\cos\theta + \frac{k}{m}x = 0$

Note that the gravitational acceleration g was not included since we are measuring x from the equilibrium position.

In the tangential direction, we have

$m a_t = -mg\sin\theta$ (a_t is the tangential acceleration)

$m(l\ddot{\theta} - \ddot{x}\sin\theta) = -mg\sin\theta$

$\therefore \quad l\ddot{\theta} - \ddot{x}\sin\theta + g\sin\theta = 0$

The equations of motion are

$$\ddot{x} - l\ddot{\theta}\sin\theta - l\dot{\theta}^2\cos\theta + (k/m)x = 0$$
$$l\ddot{\theta} - \ddot{x}\sin\theta + g\sin\theta = 0$$

To find the natural frequencies we assume

$x = X\sin\omega t$ and $\theta = \Theta\sin\omega t$ .... (*)

It is clear that $\Theta = 0$, $\omega = \omega_n = \sqrt{k/m}$ is a solution which represents the vertical oscillations (no rotation).

Thus, $\left(\dfrac{\theta}{X}\right)_1 = 0$ and $w_1 = \sqrt{\dfrac{k}{m}}$

In this case, we have $\ddot{X} + \frac{k}{m} X = 0$.

Now, assuming small θ ($\sin\theta \approx \theta$ and $\cos\theta \approx 1$) the equations of motion become

$$\ddot{X} - l\ddot{\theta}\theta - l\dot{\theta}^2 + w_n^2 X = 0$$
$$l\ddot{\theta} - \ddot{X}\theta + g\theta = 0$$

Plugging in (*) in the second equation of motion, we get

$$-w^2 l\theta \sin wt + w^2 X\theta \sin^2 wt + g\theta \sin wt = 0$$
$$\therefore \quad -1 + \left(\frac{X}{l}\right) \sin wt + \left(\frac{w_{n\theta}}{w}\right)^2 = 0 \text{, where } w_{n\theta}^2 = g/l$$

if $X << l$ then $\left(\frac{X}{l}\right) \sin wt << 1$. Hence

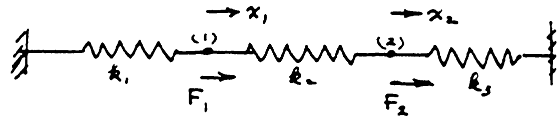
$\dfrac{w_{n\theta}}{w} = 1 \Rightarrow w = w_{n\theta} = \sqrt{g/l}$ which is the same as the frequency of a simple pendulum

The above approximations lead to pure rotations. In this case, $\left(\dfrac{\theta}{X}\right)_2 = \infty$ (very large) and $w_2 = \sqrt{g/l}$.

To get a better result one must plug in (*) in both differential equations and solve for w_2.

large $k \Rightarrow$ large w_1 and vice versa.
$\therefore$ In order for the skiers not to experience very fast oscillations k should be low.

6-1



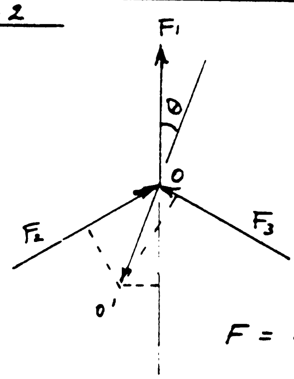
$$F_1 = -k_1 x_1 - k_2(x_1 - x_2)$$

$$F_2 = -k_3 x_2 - k_2(x_2 - x_1)$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} -(k_1+k_2) & k_2 \\ k_2 & -(k_2+k_3) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} K \end{bmatrix}^{-1} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{-1}{k_1k_2 + k_1k_3 + k_2k_3} \begin{bmatrix} (k_2+k_3) & k_2 \\ k_2 & (k_1+k_2) \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

6-2



Let o move to o' with δ small

Then $F_1 = \delta k \cos\theta$

$F_2 = \delta k \cos(60 - \theta)$

$F_3 = \delta k \cos(60 + \theta)$

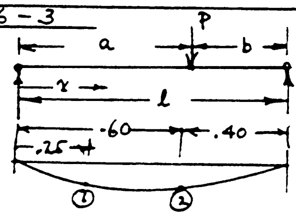
Force along δ is

$$F = F_1 \cos\theta + F_2 \cos(60-\theta) + F_3 \cos(60+\theta)$$

$$= k\delta\left[\cos^2\theta + \cos^2(60-\theta) + \cos^2(60+\theta)\right] = 1.5\, k\delta$$

$\therefore \dfrac{\delta}{F} = \dfrac{1}{1.5k}$ and independent of θ

6-3



$$y = \frac{Pbx}{6EIl}(l^2 - x^2 - b^2) \qquad x \leq a$$

Position ① $= 0.25l$, position ② $= 0.60l$

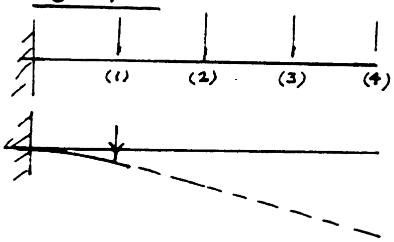
$$a_{11} = \frac{(.75)(.25)l^3}{6\,EI}(1 - .25^2 - .75^2) = 0.0114\,\frac{l^3}{EI}$$

$$a_{12} = a_{21} = \frac{(0.25)(.40)l^3}{6\,EI}(1 - .40^2 - .25^2) = 0.0130\,\frac{l^3}{EI}$$

$$a_{22} = \frac{(.40)(.60)l^3}{6\,EI}(1 - .60^2 - .40^2) = 0.0192\,\frac{l^3}{EI}$$

$$a = \frac{l^3}{EI}\begin{bmatrix} 0.0114 & 0.0130 \\ 0.0130 & 0.0192 \end{bmatrix}$$

6 - 4



$$a_{11} = \frac{\left(\frac{\ell}{4}\right)^3}{3EI} = \frac{\ell^3}{192EI}$$

$$\theta_{11} = \frac{\left(\frac{\ell}{4}\right)^2}{2EI} = \frac{\ell^2}{32EI}$$

$$a_{21} = a_{11} + \theta_{11}\frac{\ell}{4} = \frac{\ell^3}{192EI} + \frac{\ell^3}{128EI} = \frac{2.50\,\ell^3}{192EI}$$

$$a_{31} = a_{21} + \frac{\ell^3}{128EI} = \frac{4\,\ell^3}{192EI}$$

$$a_{41} = a_{31} + \frac{\ell^3}{128EI} = \frac{5.50\,\ell^3}{192EI}$$

Place unit load at ② and repeat

$$a_{22} = \frac{\left(\frac{\ell}{2}\right)^3}{3EI} = \frac{\ell^3}{24EI}$$

$$\theta_{22} = \frac{\left(\frac{\ell}{2}\right)^2}{2EI} = \frac{\ell^2}{8EI}$$

$$a_{32} = a_{22} + \theta_{22}\frac{\ell}{4} = \left(\frac{1}{24} + \frac{1}{32}\right)\frac{\ell^3}{EI} = \frac{7}{96}\frac{\ell^3}{EI}$$

$$a_{42} = \left(\frac{7}{96} + \frac{1}{32}\right)\frac{\ell^3}{EI} = \frac{10}{96}\frac{\ell^3}{EI}$$

$$a_{33} = \frac{\left(\frac{3}{4}\ell\right)^3}{3EI} = \frac{9}{64}\frac{\ell^3}{EI}, \qquad \theta_{33} = \frac{\left(\frac{3}{4}\ell\right)^2}{2EI} = \frac{9}{32}\frac{\ell^2}{EI}$$

$$a_{43} = a_{33} + \theta_{33}\frac{\ell}{4} = \left(\frac{9}{64} + \frac{9}{128}\right)\frac{\ell^3}{EI} = \frac{27}{128}\frac{\ell^3}{EI}$$

$$a_{44} = \frac{\ell^3}{3EI}$$

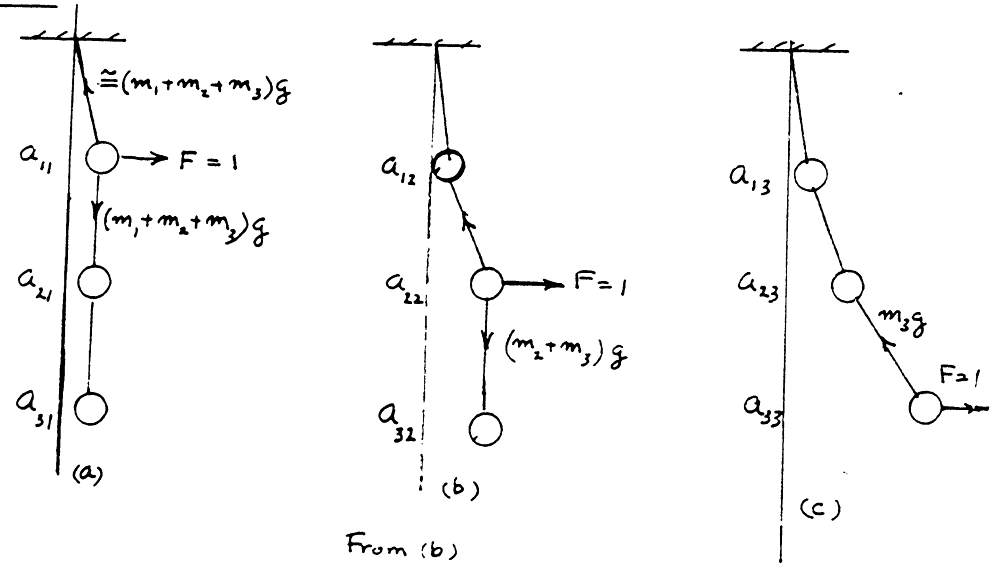
$$a = \frac{\ell^3}{EI}\begin{bmatrix} \frac{1}{192} & \frac{2.5}{192} & \frac{4}{192} & \frac{5.5}{192} \\ \frac{2.5}{192} & \frac{1}{24} & \frac{7}{96} & \frac{10}{96} \\ \frac{4.0}{192} & \frac{7}{96} & \frac{9}{64} & \frac{27}{128} \\ \frac{5.5}{192} & \frac{10}{96} & \frac{27}{128} & \frac{1}{3} \end{bmatrix} = \frac{\ell^3}{192EI}\begin{bmatrix} 1 & 2.5 & 4.0 & 5.5 \\ 2.5 & 8.0 & 14.0 & 20.0 \\ 4.0 & 14.0 & 27.0 & 40.5 \\ 5.5 & 20.0 & 40.5 & 64.0 \end{bmatrix}$$

```
EDU»inv(a)

ans =

    6.2680   -3.9381    1.4845   -0.2474
   -3.9381    4.7835   -3.1959    0.8660
    1.4845   -3.1959    3.2990   -1.2165
   -0.2474    0.8660   -1.2165    0.5361
```

6-5



From (a)

$$\frac{a_{11}}{\ell_1}(m_1+m_2+m_3)g = 1$$

$$a_{11} = a_{21} = a_{31} = \frac{1}{m_1+m_2+m_3}\cdot\frac{\ell_1}{g}$$

From (b)

$$\left(\frac{a_{22}-a_{12}}{\ell_2}\right)(m_2+m_3)g = 1$$

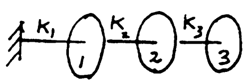
$$a_{22} = \frac{1}{m_2+m_3}\frac{\ell_2}{g} + \frac{1}{m_1+m_2+m_3}\frac{\ell_1}{g} = a_{32}$$

From (c)

$$\left(\frac{a_{33}-a_{23}}{\ell_3}\right)m_3 g = 1 \qquad a_{33} = \frac{1}{m_3}\frac{\ell_3}{g} + \frac{1}{m_2+m_3}\frac{\ell_2}{g} + \frac{1}{m_1+m_2+m_3}\frac{\ell_1}{g}$$

$$a_{ij} = a_{ji}$$

6-6



Give each disk a unit rotation holding other disks with zero rotation. Torque required is then:

$\theta_1 = 1.0 \qquad T_1 = (K_1+K_2), \quad T_0 = -K_1, \quad T_2 = -K_2, \quad T_3 = 0$

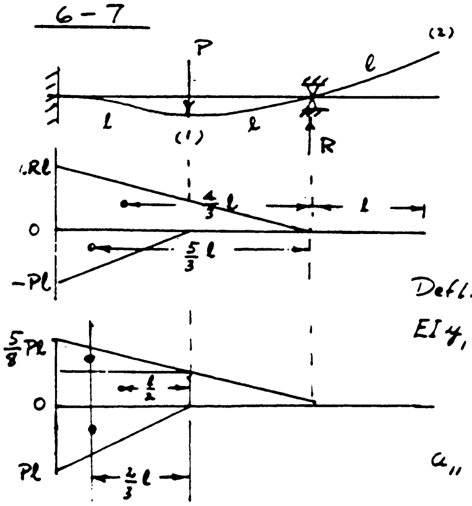
$\theta_2 = 1.0 \qquad T_1 = -K_2, \quad T_2 = (K_2+K_3), \quad T_3 = -K_3$

$\theta_3 = 1.0 \qquad T_1 = 0, \quad T_2 = -K_3, \quad T_3 = K_3$

$$[K] = \begin{bmatrix} (K_1+K_2) & -K_2 & 0 \\ -K_2 & (K_2+K_3) & -K_3 \\ 0 & -K_3 & K_3 \end{bmatrix} \qquad [a] = \begin{bmatrix} 1/K_1 & 1/K_1 & 1/K_1 \\ 1/K_1 & (1/K_1+1/K_2) & (1/K_1+1/K_2) \\ 1/K_1 & (1/K_1+1/K_2) & (1/K_1+1/K_2+1/K_3) \end{bmatrix}$$

For flexibility apply unit torque to each disk in turn and measure rotation.

6-7



Defl. at R = 0

$$\frac{1}{2}(2Rl\times 2l)\frac{4}{3}l - \frac{1}{2}(Pl\times l)\frac{5}{3}l = 0$$

$$\therefore R = \frac{5}{16}P$$

$$2Rl = \frac{5}{8}Pl$$

Defl. at (1) = moment of area about (1)

$$EIy_1 = \left(\frac{5}{16}Pl\times l\right)\frac{l}{2} + \frac{1}{2}\left(\frac{5}{16}Pl^2\right)\frac{2}{3}l - \frac{1}{2}\left(Pl^2\right)\frac{2}{3}l$$

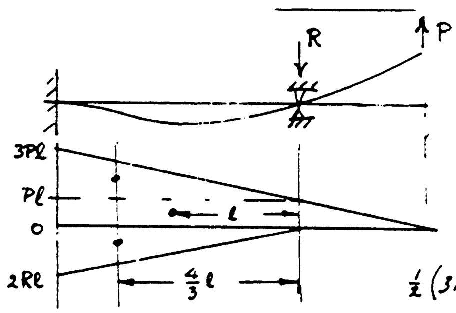
$$= Pl^3\left(\frac{5}{32}+\frac{5}{48}-\frac{1}{3}\right) = -\frac{3.5}{48}Pl^3$$

$$a_{11} = \frac{3.50}{48}\frac{l^3}{EI} = \frac{7}{96}\frac{l^3}{EI}$$

Defl. at (2) = moment of area about (2)

$$EIy_2 = \frac{1}{2}\left[\frac{5}{8}Pl\times 2l\right]\frac{7}{3}l - \frac{1}{2}\left[Pl^2\right]\frac{8}{3}l = Pl^3\left(\frac{35}{24}-\frac{8}{6}\right) = \frac{1}{8}Pl^3$$

$$\therefore a_{21} = a_{12} = \frac{1}{8}\frac{l^3}{EI}$$



Defl at R = 0

$$(2Pl^2)l + \frac{1}{2}(2Pl\times 2l)\frac{4}{3}l - \frac{1}{2}(2Rl\times 2l)\frac{4}{3}l = 0$$

$$Pl^3\left(2+\frac{8}{3}\right) = Rl^3\left(\frac{8}{3}\right) \quad \therefore R = \frac{7}{4}P$$

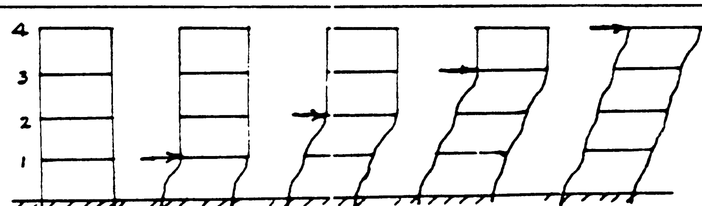
For a_{22}

$$\frac{1}{2}(3Pl\times 3l)2l - \frac{1}{2}\left(\frac{7}{2}Pl\times 2l\right)\left(l+\frac{4}{3}l\right) = EIy_2$$

$$Pl^3(9-8.1666) = \left(\frac{54}{6}-\frac{49}{6}\right)Pl^3 = \frac{5}{6}Pl^3 \qquad \therefore a_{22} = \frac{5}{6}\frac{l^3}{EI}$$

$$[a] = \frac{l^3}{EI}\begin{bmatrix} 7/96 & 1/8 \\ 1/8 & 5/6 \end{bmatrix}$$

6-8



$a_{11} = \frac{1}{12}\frac{l^3}{EI}$ $\quad a_{12} = \frac{1}{12}\frac{l^3}{EI}$ $\quad a_{13} = \frac{1}{12}\frac{l^3}{EI}$ $\quad a_{14} = \frac{1}{12}\frac{l^3}{EI}$

$a_{21} = \frac{1}{12}$ " $\quad a_{22} = \frac{2}{12}$ " $\quad a_{23} = \frac{2}{12}$ " $\quad a_{24} = \frac{2}{12}$ "

$a_{31} = \frac{1}{12}$ " $\quad a_{32} = \frac{2}{12}$ " $\quad a_{33} = \frac{3}{12}$ " $\quad a_{34} = \frac{3}{12}$ "

$a_{41} = \frac{1}{12}$ " $\quad a_{42} = \frac{2}{12}$ " $\quad a_{43} = \frac{3}{12}$ " $\quad a_{44} = \frac{4}{12}$ "

$$[a] = \frac{l^3}{12EI}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

6-8 Cont.

$$[k] = [a]^{-1}$$

$$D = \det.[a] = \begin{vmatrix} 2 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 4 \end{vmatrix} - \begin{vmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{vmatrix} - \begin{vmatrix} 1 & 2 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= 2(12-9) - 1(12-9) + 1(8-6) - 1(0) - 2(8-6) + 2(4-3)$$
$$- 2(4-3) + 0 + 0 - 0 + 0 - 0 = 1$$

$$C_{11} = \begin{vmatrix} 2 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 4 \end{vmatrix} \quad C_{12} = -\begin{vmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{vmatrix} \quad C_{13} = \begin{vmatrix} 1 & 2 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{vmatrix} = 0 \quad C_{14} = 0$$

$$C_{21} = -\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 2 & 3 & 4 \end{vmatrix} \quad C_{22} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{vmatrix} \quad C_{23} = -\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{vmatrix} \quad C_{24} = 0$$

$$C_{31} = 0 \quad C_{32} = -\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 4 \end{vmatrix} \quad C_{33} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 4 \end{vmatrix} \quad C_{34} = -\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix}$$

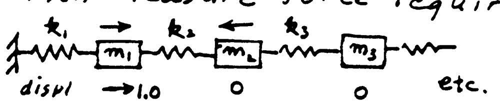
$$C_{41} = 0 \quad C_{42} = 0 \quad C_{43} = -\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix} \quad C_{44} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix}$$

$$C_{11}' = C_{11} = 2, \quad C_{12}' = -1 \quad C_{21}' = -1, \quad C_{22}' = 2 \quad C_{23}' = -1$$

$$C_{32}' = -1 \quad C_{33}' = 2 \quad C_{34}' = -1 \quad C_{43}' = -1 \quad C_{44}' = 1$$

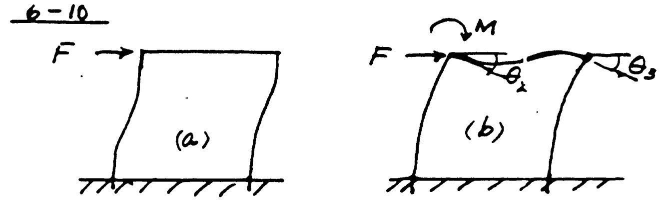
$$\therefore [k] = \frac{12EI}{l^3} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

6-9 Give each mass a unit displ. keeping all others at zero. Then measure force required. ie for $x_1 = 1.0$



$$[k] = \begin{bmatrix} (k_1+k_2) & -k_2 & 0 & 0 & 0 & \cdots \\ -k_2 & (k_2+k_3) & -k_3 & 0 & 0 & \cdots \\ 0 & -k_3 & (k_3+k_4) & -k_4 & 0 & \cdots \\ 0 & 0 & -k_4 & (k_4+k_5) & -k_5 & \cdots \\ \vdots & & & & & \end{bmatrix}$$

6-10



(a) $k = 2 \times \frac{12EI}{\ell^3} = 24\frac{EI}{\ell^3}$

(b) $F = \frac{EI}{\ell^3}\left[24 - 6\,\theta_2\ell - 6\,\theta_3\ell\right]$ $\qquad \theta_2 = \theta_3 = \theta$

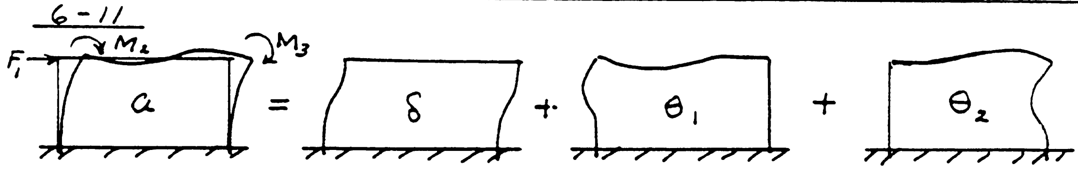
$M = \frac{EI}{\ell^2}\left[-6 + 8\,\theta_2\ell + 2\theta_3\ell\right] = 0$ $\qquad$ (see Prob 6-11)

$-6 + 10\,\theta\ell = 0 \quad \therefore \ \theta = \frac{6}{10\ell}$

$F = \frac{EI}{\ell^3}\left[24 - 12 \times \frac{6}{10}\right] = 16.8\frac{EI}{\ell^3}$

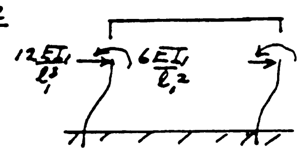
Ratio $\frac{(a)}{(b)} = \frac{24}{16.8}$

6-11

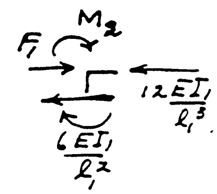


$\underline{a}$ is superposition of $\delta + \theta_1 + \theta_2$

$\underline{\delta \text{ mode}}$

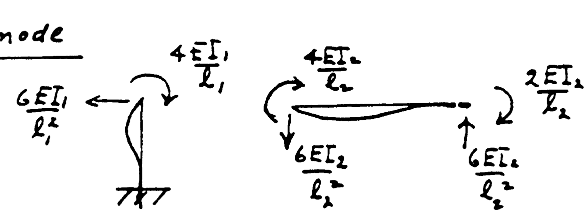


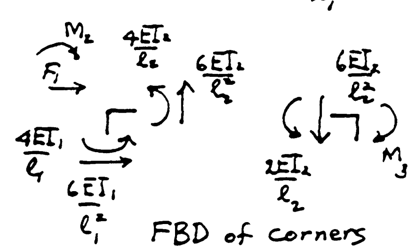
FBD of corner



$\therefore F_1 = 24\frac{EI_1}{\ell_1^3}\delta \qquad M_2 = M_3 = -6\frac{EI_1}{\ell_1^2}\delta$

$\underline{\theta_1 \text{ mode}}$



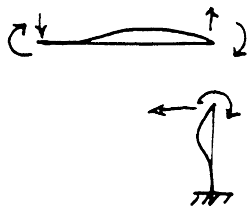


FBD of corners

$F_1 = -\frac{6EI_1}{\ell_1^2}\theta_1, \qquad M_2 = \left(\frac{4EI_1}{\ell_1} + \frac{4EI_2}{\ell_2}\right)\theta_1, \qquad M_3 = \left(\frac{2EI_2}{\ell_2}\right)\theta_1$

6-11 Cont

θ_2 mode

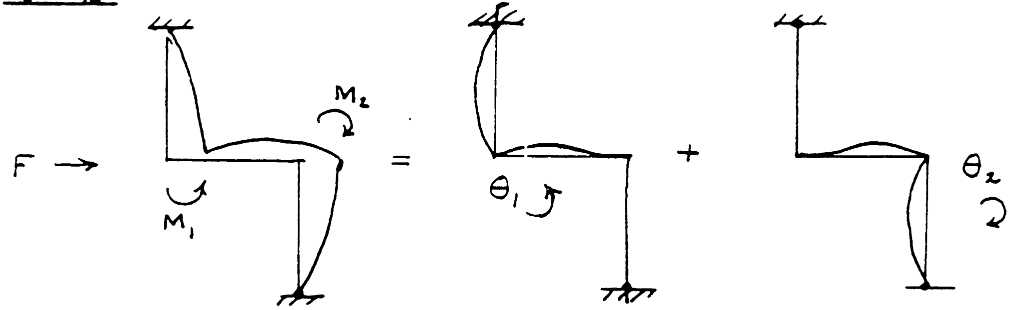


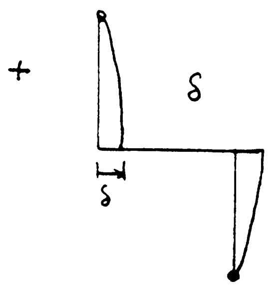
Same as mode θ_1 but shifted to right corner

Sum results for δ, θ_1 & θ_2

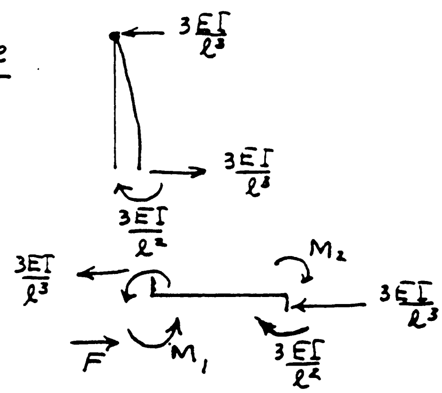
$$\begin{Bmatrix} F_1 \\ M_1 \\ M_2 \end{Bmatrix} = \begin{bmatrix} 24\frac{EI_1}{\ell_1^3} & -6\frac{EI_1}{\ell_1^2} & -6\frac{EI_1}{\ell_1^2} \\ -6\frac{EI_1}{\ell_1^2} & \left(\frac{4EI_1}{\ell_1}+\frac{4EI_2}{\ell_2}\right) & 2\frac{EI_2}{\ell_2} \\ -6\frac{EI_1}{\ell_1^2} & \frac{2EI_2}{\ell_2} & \left(\frac{4EI_1}{\ell_1}+\frac{4EI_2}{\ell_2}\right) \end{bmatrix} \begin{Bmatrix} \delta \\ \theta_1 \\ \theta_2 \end{Bmatrix}$$

6-12





δ-mode



$$F = 6\frac{EI}{\ell^3}\delta$$

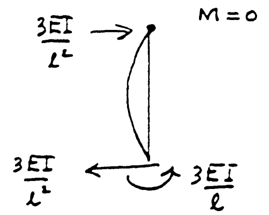
$$M_1 = -3\frac{EI}{\ell^2}\delta$$

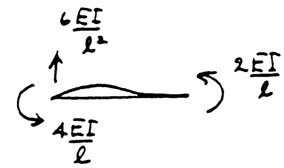
$$M_2 = -\frac{3EI}{\ell^2}\delta$$

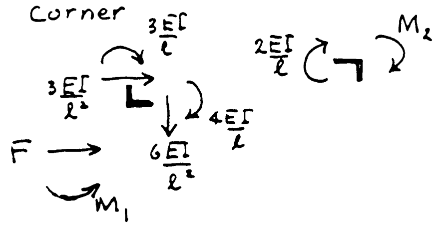
from corners

6-12 Cont

θ_1-mode







$$F = -\frac{3EI}{\ell^2}\theta_1$$

$$M_1 = \left(\frac{3EI}{\ell} + \frac{4EI}{\ell}\right)(\theta_1)$$

$$M_2 = -\frac{2EI}{\ell}(\theta_1)$$

θ_2-mode this mode is same as above moved to right corner with θ_1 replaced by $+\theta_2$

Add results

$$\begin{Bmatrix} F \\ M_1 \\ M_2 \end{Bmatrix} = \begin{bmatrix} \frac{6EI}{\ell^3} & -\frac{3EI}{\ell^2} & -\frac{3EI}{\ell^2} \\ -\frac{3EI}{\ell^2} & \left(\frac{3EI}{\ell} + \frac{4EI}{\ell}\right) & -\frac{2EI}{\ell} \\ -\frac{3EI}{\ell^2} & -\frac{2EI}{\ell} & \left(\frac{3EI}{\ell} + \frac{4EI}{\ell}\right) \end{bmatrix} \begin{Bmatrix} \delta \\ \theta_1 \\ \theta_2 \end{Bmatrix}$$

Since no moments were applied at corners $M_1 = M_2 = 0$

$$\left.\begin{aligned} 0 &= -\frac{3EI}{\ell^2}\delta + \frac{7EI}{\ell}\theta_1 - \frac{2EI}{\ell}\theta_2 \\ 0 &= -\frac{3EI}{\ell^2}\delta - \frac{2EI}{\ell}\theta_1 + \frac{7EI}{\ell}\theta_2 \end{aligned}\right\}$$

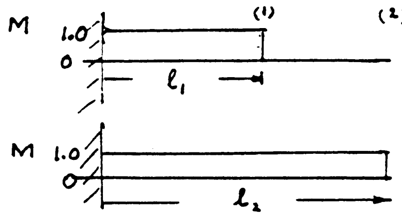
Subtr. we find $\theta_1 = \theta_2 = \theta$ as expected by symmetry

$$M_1 = -\frac{3EI}{\ell^2}\delta + \frac{5EI}{\ell}\theta = 0 \qquad \therefore\ \theta = \frac{3\delta}{5\ell}$$

$$F = \frac{6EI}{\ell^3}\delta - \frac{6EI}{\ell^2}\left(\frac{3\delta}{5\ell}\right) = \underline{\underline{2.40\,\frac{EI}{\ell^3}\,\delta}}$$

6-13

Moment Diagram



$$EI\,\theta_{21} = EI\,\theta_{12} = M_1 \ell_1 = M_2 \ell_2$$

$$M_1 = M_2 = 1$$

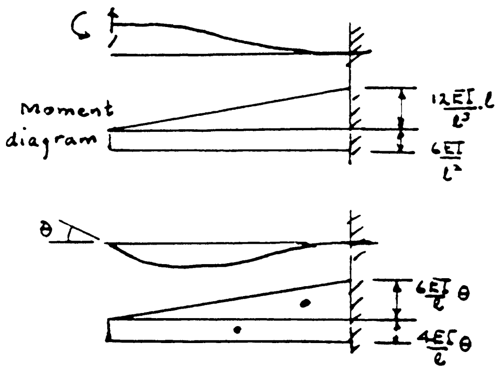
$$\therefore \quad \theta_{12} = \theta_{21}$$

By Work done

M_1 1st followed by M_2 $\qquad W = \frac{1}{2}M_1\theta_{11} + \frac{1}{2}M_2\theta_{22} + M_1\theta_{12}$

M_2 1st " " M_1 $\qquad W = \frac{1}{2}M_2\theta_{22} + \frac{1}{2}M_1\theta_{11} + M_2\theta_{21}$

6-14



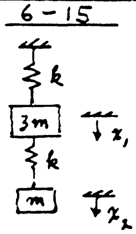
Difference in slope $= 0$

$$\left(\frac{1}{2}\cdot\frac{12EI}{\ell^2}\cdot\ell\right) - \left(\frac{6EI}{\ell^2}\cdot\ell\right) = 0$$

Difference in defl. $= 0$

$$\left(\frac{1}{2}\,\frac{6EI}{\ell}\right)\frac{2}{3}\ell - \left(\frac{4EI}{\ell}\cdot\ell\right)\frac{\ell}{2} = 0$$

6-15



$$m\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + k\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \{0\} \qquad \text{Let } \lambda = \frac{m\omega^2}{k}$$

$$\begin{vmatrix} (2-3\lambda) & -1 \\ -1 & (1-\lambda) \end{vmatrix} = 0 \qquad \lambda^2 - \frac{5}{3}\lambda + \frac{1}{3} = 0$$

$$\lambda = 0.8333 \pm \sqrt{.6944 - .3333}$$

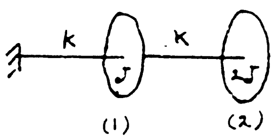
$$= 0.8333 \pm 0.6010 = \begin{cases} 0.2323 \\ 1.4343 \end{cases}$$

$$\text{Adj. matrix} = \begin{bmatrix} (1-\lambda) & 1 \\ 1 & (2-3\lambda) \end{bmatrix}$$

Subst. λ_1 into either column for 1st mode $= \begin{Bmatrix} .7677 \\ 1.000 \end{Bmatrix}$

Subst λ_2 into " " " 2nd mode $= \begin{Bmatrix} -.4343 \\ 1.000 \end{Bmatrix}$

6-16



$$\begin{bmatrix} J & 0 \\ 0 & 2J \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} 2K & -K \\ -K & K \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \{0\}$$

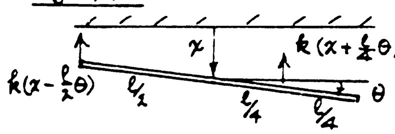
Let $\lambda = \dfrac{\omega^2 J}{K}$

$$\begin{vmatrix} (2-\lambda) & -1 \\ -1 & (1-2\lambda) \end{vmatrix} = 0 \qquad 2\lambda^2 - 5\lambda + 1 = 0$$

$$\lambda = \frac{5}{4} \pm \sqrt{\frac{25}{16} - \frac{8}{16}} = \begin{cases} 0.2192 \\ 2.2808 \end{cases}$$

$$\text{Adj. matrix} = \begin{bmatrix} (1-2\lambda) & 1 \\ 1 & (2-\lambda) \end{bmatrix} \qquad \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}_1 = \begin{Bmatrix} 1.000 \\ 1.781 \end{Bmatrix} \qquad \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}_2 = \begin{Bmatrix} 1.00 \\ -.2808 \end{Bmatrix}$$

6-17



$$m\frac{\ell^2}{12}\ddot{\theta} = \frac{\ell}{2}k(x - \tfrac{\ell}{2}\theta) - \frac{\ell}{4}k(x + \tfrac{\ell}{4}\theta) = \frac{k\ell}{4}x - \frac{5\ell^2}{16}k\theta$$

$$m\ddot{x} = -k(x - \tfrac{\ell}{2}\theta) - k(x + \tfrac{\ell}{4}\theta) = -2kx + \frac{k\ell}{4}\theta$$

$$m\begin{bmatrix} \frac{\ell^2}{12} & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{\theta} \\ \ddot{x} \end{Bmatrix} + k\begin{bmatrix} \frac{5}{16}\ell^2 & -\frac{\ell}{4} \\ -\frac{\ell}{4} & 2 \end{bmatrix}\begin{Bmatrix} \theta \\ x \end{Bmatrix} = \{0\} \qquad \text{Let } \lambda = \frac{\omega^2 m}{k}$$

$$\begin{vmatrix} (\frac{5}{16}\ell^2 - \frac{\ell^2}{12}\lambda) & -\frac{\ell}{4} \\ -\frac{\ell}{4} & (2-\lambda) \end{vmatrix} = 0 \qquad \lambda^2 - \frac{23}{4}\lambda + \frac{9\times 12}{16} = 0$$

$$\lambda = \frac{23}{8} \pm \sqrt{\frac{97}{16}}$$

$$\lambda = \frac{1}{8}(23 \pm 9.849) = \begin{cases} 1.644 \\ 4.106 \end{cases} \qquad \frac{\ell\theta}{x} = 4(2-\lambda) = \begin{cases} 1.424 \\ -8.424 \end{cases}$$

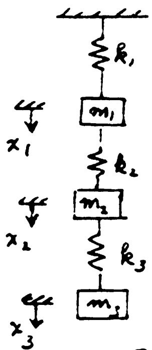
$$\begin{Bmatrix} \theta \\ x \end{Bmatrix}_1 = \begin{Bmatrix} 1.424/\ell \\ 1.000 \end{Bmatrix} \qquad \begin{Bmatrix} \theta \\ x \end{Bmatrix}_2 = \begin{Bmatrix} -8.424/\ell \\ 1.000 \end{Bmatrix}$$

$$P = \begin{bmatrix} 1.424/\ell & -8.424/\ell \\ 1.000 & 1.000 \end{bmatrix}$$ To show that $P'KP$ = diagonal

$$\begin{bmatrix} 1.424/\ell & 1.0 \\ -8.424/\ell & 1.0 \end{bmatrix}\begin{bmatrix} \frac{5}{16}\ell^2 & -\frac{\ell}{4} \\ -\frac{\ell}{4} & 2 \end{bmatrix}\begin{bmatrix} 1.424/\ell & -8.424/\ell \\ 1.000 & 1.000 \end{bmatrix} = \begin{bmatrix} 1.424/\ell & 1.0 \\ -8.424/\ell & 1.0 \end{bmatrix}\begin{bmatrix} .195\ell & -2.8825 \\ 1.644 & 4.106 \end{bmatrix}$$

$$= \begin{bmatrix} 1.9217 & 0.0013 \\ 0.0013 & 28.388 \end{bmatrix} \cong \begin{bmatrix} 1.9217 & 0 \\ 0 & 28.388 \end{bmatrix}$$

6-18



Place unit load at 1.

$$a_{11} = \frac{1}{k_1} = a_{21} = a_{31} = a_{12} = a_{13}$$

Place unit load at 2

$$a_{22} = \left(\frac{1}{k_1} + \frac{1}{k_2}\right) = a_{32} = a_{23}$$

Place unit load at 3

$$a_{33} = \left(\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}\right)$$

Eq. of motion:

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \omega^2 \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

6-19

$$\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \{0\} \qquad \text{Let } \lambda = \frac{m\omega^2}{k}$$

$$\begin{vmatrix} (2-\lambda) & -1 \\ -1 & (2-2\lambda) \end{vmatrix} = 0 \qquad \lambda^2 - 3\lambda + \frac{3}{2} = 0$$

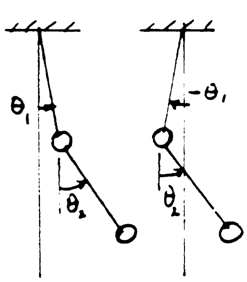
$$\lambda = \frac{3}{2} \pm \sqrt{\frac{9}{4} - \frac{3}{2}} = 1.5 \pm 0.866 = \begin{cases} 0.634 \\ 2.366 \end{cases}$$

$$\frac{x_1}{x_2} = \frac{2(1-\lambda)}{1} = \begin{cases} 0.732 \\ -2.732 \end{cases} \qquad \therefore P = \begin{bmatrix} 0.732 & -2.732 \\ 1.000 & 1.000 \end{bmatrix}$$

$$P'MP\ddot{y} + P'KPy = 0$$

$$\begin{bmatrix} 2.535 & 0 \\ 0 & 9.48 \end{bmatrix} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} + \begin{bmatrix} 1.606 & 0 \\ 0 & 22.33 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \{0\} \qquad \text{uncoupled.}$$

6-20



$$\frac{l}{g} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = 0$$

$\therefore$ dynamically coupled.

From Prob. 5-10

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}_1 = \begin{Bmatrix} 0.707 \\ 1.00 \end{Bmatrix} \qquad \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}_2 = \begin{Bmatrix} -0.707 \\ 1.00 \end{Bmatrix}$$

6-20 Cont.

Gen. mass

$$\text{mode } 1 \quad (.707 \quad 1.0)\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} .707 \\ 1.0 \end{Bmatrix} = 3.414$$

$$\text{mode } 2 \quad (-.707 \quad 1.0)\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} -.707 \\ 1.00 \end{Bmatrix} = 0.586$$

$$P = \begin{bmatrix} .707 & -.707 \\ 1.0 & 1.0 \end{bmatrix} \qquad \tilde{P} = \begin{bmatrix} .207 & -1.208 \\ .293 & 1.707 \end{bmatrix}$$

$$\tilde{P}'M\tilde{P} = \begin{bmatrix} .207 & .293 \\ -1.207 & 1.707 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} .207 & -1.207 \\ .293 & 1.707 \end{bmatrix} = \begin{bmatrix} 0.293 & 0 \\ 0 & 1.707 \end{bmatrix}$$

$$\tilde{P}'K\tilde{P} = \begin{bmatrix} & '' & \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} & '' & \end{bmatrix} = \begin{bmatrix} .1714 & 0 \\ 0 & 5.82 \end{bmatrix}$$

Normalized eq. of motion becomes

$$\begin{bmatrix} .293 & 0 \\ 0 & 1.707 \end{bmatrix}\begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} + \begin{bmatrix} .1714 & 0 \\ 0 & 5.82 \end{bmatrix}\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix} \quad \therefore \text{ uncoupled}$$

6-21 See Prob. 6-11

$$\begin{bmatrix} (m_1+m_2) & 0 & 0 \\ 0 & J_1 & 0 \\ 0 & 0 & J_2 \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} \frac{24EI_1}{\ell_1^3} & -\frac{6EI_1}{\ell_1^2} & -\frac{6EI_1}{\ell_1^2} \\ -\frac{6EI_1}{\ell_1^2} & \left(\frac{4EI_1}{\ell_1}+\frac{4EI_2}{\ell_2}\right) & \frac{2EI_2}{\ell_2} \\ -\frac{6EI_1}{\ell_1^2} & \frac{2EI_2}{\ell_2} & \left(\frac{4EI_1}{\ell_1}+\frac{4EI_2}{\ell_2}\right) \end{bmatrix}\begin{Bmatrix} x \\ \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix}$$

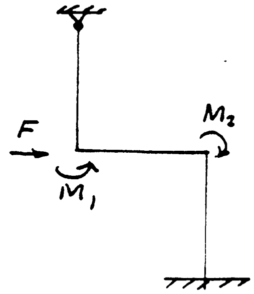
Assign numbers before solving.

6-22 See Prob 6-12

$$\begin{bmatrix} (m_1+m_2) & 0 & 0 \\ 0 & J_1 & 0 \\ 0 & 0 & J_2 \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} \frac{6EI}{\ell^3} & -\frac{3EI}{\ell^2} & -\frac{3EI}{\ell^2} \\ -\frac{3EI}{\ell^2} & \frac{7EI}{\ell} & -\frac{2EI}{\ell} \\ -\frac{3EI}{\ell^2} & -\frac{2EI}{\ell} & \frac{7EI}{\ell} \end{bmatrix}\begin{Bmatrix} x \\ \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix}$$

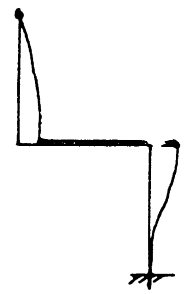
Assign numbers before solving

6-23

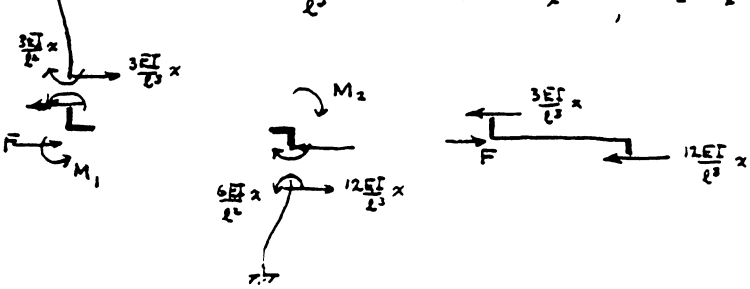


The problem is the superposition of the three modes below. Use Figure 6.4-2 and examine the free-body-diagrams of members and corners.

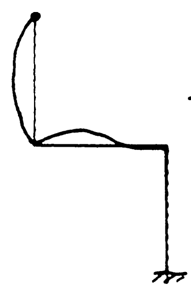
x-mode



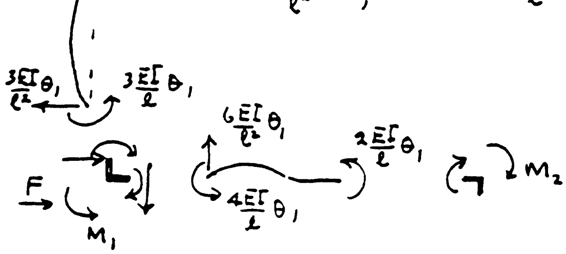
$F = \frac{15EI}{\ell^3}x \qquad M_1 = -\frac{3EI}{\ell^2}x, \qquad M_2 = \frac{6EI}{\ell^2}x$



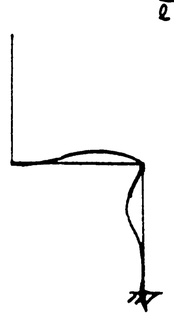
θ_1-mode

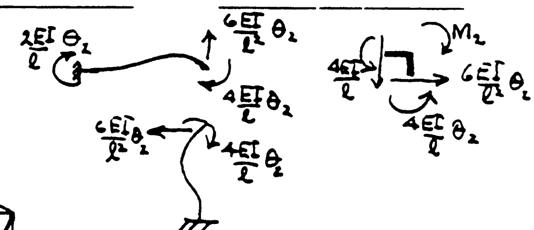


$F = -\frac{3EI}{\ell^2}\theta_1, \qquad M_1 = \frac{7EI}{\ell}\theta_1 \qquad M_2 = -\frac{2EI}{\ell}\theta_1$



θ_2-mode





$$\begin{Bmatrix} F \\ M_1 \\ M_2 \end{Bmatrix} = \begin{bmatrix} \frac{15EI}{\ell^3} & -\frac{3EI}{\ell^2} & \frac{6EI}{\ell^2} \\ -\frac{3EI}{\ell^2} & \frac{7EI}{\ell} & -\frac{2EI}{\ell} \\ \frac{6EI}{\ell^2} & -\frac{2EI}{\ell} & \frac{8EI}{\ell} \end{bmatrix} \begin{Bmatrix} x \\ \theta_1 \\ \theta_2 \end{Bmatrix}$$

6-24

$$m\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{Bmatrix}\ddot{x}_1\\\ddot{x}_2\end{Bmatrix} + c\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}\begin{Bmatrix}\dot{x}_1\\\dot{x}_2\end{Bmatrix} + k\begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}\begin{Bmatrix}x_1\\x_2\end{Bmatrix} = \begin{Bmatrix}F_0\\0\end{Bmatrix}\sin\omega t$$

$\therefore$ Damping is not proportional

6-25

char. eq. $\begin{vmatrix}(2-\lambda) & -1\\-1 & (2-\lambda)\end{vmatrix} = 0 \qquad \lambda = \begin{cases}1\\3\end{cases} \qquad \begin{Bmatrix}x_1\\x_2\end{Bmatrix}_1 = \begin{Bmatrix}1\\1\end{Bmatrix}$

$\lambda = \dfrac{\omega^2 m}{k}$

$$\begin{Bmatrix}x_1\\x_2\end{Bmatrix}_2 = \begin{Bmatrix}-1\\1\end{Bmatrix}$$

$$\therefore\ P = \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix} \qquad \tilde{P} = \frac{1}{\sqrt{2m}}\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}$$

Let $x = \tilde{P}y$ in eq. Prob. 6-24 and premult. by $\tilde{P}'$

$$\tilde{P}'M\tilde{P}\ddot{y} + \tilde{P}'C\tilde{P}\dot{y} + \tilde{P}'k\tilde{P}y = \tilde{P}'F$$

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{Bmatrix}\ddot{y}_1\\\ddot{y}_2\end{Bmatrix} + \frac{c}{2m}\begin{bmatrix}1 & -1\\-1 & 5\end{bmatrix}\begin{Bmatrix}\dot{y}_1\\\dot{y}_2\end{Bmatrix} + \frac{k}{m}\begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}\begin{Bmatrix}y_1\\y_2\end{Bmatrix} = \begin{Bmatrix}F_0\\-F_0\end{Bmatrix}\frac{\sin\omega t}{\sqrt{2m}}$$

$\therefore$ coupled only by damping

$$\ddot{y}_1 + \frac{c}{2m}(\dot{y}_1 - \dot{y}_2) + \frac{k}{m}y_1 = \frac{F_0}{\sqrt{2m}}\sin\omega t$$

$$\ddot{y}_2 + \frac{c}{2m}(-\dot{y}_1 + 5\dot{y}_2) + 3\frac{k}{m}y_2 = \frac{-F_0}{\sqrt{2m}}\sin\omega t$$

Steady State solution

$$\left(\frac{k}{m} - \omega^2 + i\frac{c}{2m}\omega\right)Y_1 - i\left(\frac{c}{2m}\omega\right)Y_2 = \frac{F_0}{\sqrt{2m}}$$

$$-i\left(\frac{c}{2m}\omega\right)Y_1 + \left(\frac{3k}{m} - \omega^2 + i\frac{5c}{2m}\omega\right)Y_2 = \frac{-F_0}{\sqrt{2m}}$$

$$Y_1 = \frac{\begin{vmatrix}1 & i\omega\frac{c}{2m}\\-1 & \left(\frac{3k}{m} - \omega^2 + i\omega\frac{5c}{2m}\right)\end{vmatrix}\dfrac{F_0}{\sqrt{2m}}}{\left(\frac{k}{m} - \omega^2 + i\frac{c}{2m}\omega\right)\left(\frac{3k}{m} - \omega^2 + i\omega\frac{5c}{2m}\right) + \left(\frac{\omega c}{2m}\right)^2}$$

6-26

Let. $\omega_0^2 = \frac{k}{m}$, $\alpha = \frac{k_1}{c}$, $\beta = \frac{k_1}{m}$

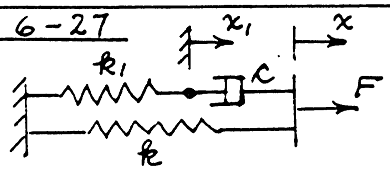
The equations are rewritten as

$$\ddot{x} = -\omega_0^2 x - \beta x_1 + \frac{F}{m}$$

$$\dot{x}_1 = \dot{x} - \alpha x_1$$

Let
$$\left.\begin{aligned} x_1 &= z_1 \\ \dot{x}_1 &= \dot{z}_1 \\ x &= z_2 \\ \dot{x} &= z_3 = \dot{z}_2 \end{aligned}\right\} \text{then} \begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{Bmatrix} = \begin{bmatrix} -\alpha & 0 & 1 \\ 0 & 0 & 1 \\ -\beta & -\omega_0^2 & 0 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \frac{F}{m} \end{Bmatrix}$$

6-27



$$F = kx + c(\dot{x} - \dot{x}_1) \qquad (a)$$

$$k_1 x_1 = c(\dot{x} - \dot{x}_1) \qquad (b)$$

Assume F to be harmonic; then from (b)

$$x_1 = \frac{i\omega c}{k_1 + i\omega c}\, x = \frac{i(\omega c/k_1)}{1 + i(\omega c/k_1)}\, x$$

Subst. into (a)

$$F = kx + i\omega c\left[1 - \frac{i(\omega c/k_1)}{1 + i(\omega c/k_1)}\right] x$$

$$= \frac{\left[k\left(1 + \frac{i\omega c}{k_1}\right) + i\omega c\right]}{\left(1 + \frac{i\omega c}{k_1}\right)} \cdot \frac{\left(1 - \frac{i\omega c}{k_1}\right)}{\left(1 - \frac{i\omega c}{k_1}\right)}\, x$$

$$= \left\{ \frac{k + (k + k_1)\left(\frac{\omega c}{k_1}\right)^2}{1 + \left(\frac{\omega c}{k_1}\right)^2} + \frac{i\omega c}{1 + \left(\frac{\omega c}{k_1}\right)^2} \right\} x$$

$$= \left\{ k_{eq} + i\omega\, c_{eq} \right\} x$$

$$= \frac{k + (k + k_1)\left(\frac{\omega c}{k_1}\right)^2}{1 + \left(\frac{\omega c}{k_1}\right)^2} \left\{ 1 + i \frac{\omega c}{k + (k + k_1)\left(\frac{\omega c}{k_1}\right)^2} \right\} x = k^*(1 + i\gamma)x$$

<u>6-28</u> Refer to Prob. 6-16

$$X_1 = \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}_1 = \begin{Bmatrix} 1.00 \\ 1.781 \end{Bmatrix} \qquad X_2 = \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}_2 = \begin{Bmatrix} 1.00 \\ -0.2808 \end{Bmatrix}$$

$$X_1' K X_2 = (1.00 \quad 1.781)\, k \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 1.00 \\ -0.2808 \end{Bmatrix}$$

$$= (1.00 \quad 1.781) \begin{Bmatrix} 2.2808 \\ -1.2808 \end{Bmatrix} = 2.2808 - 2.2811$$

$$= -0.0003 \cong 0$$

<u>6-29</u> ϕ = normal mode

$$K\phi_s = \omega_s^2 M \phi_s \qquad \times \text{ by } KM^{-1}$$

$$KM^{-1}K\phi_s = \omega_s^2 KM^{-1}M\phi_s = \omega_s^2 K\phi_s$$

$$\underline{\underline{\phi_r'[KM^{-1}K]\phi_s}} = \omega_s^2[\phi_r' K \phi_s] = \underline{\underline{0}} \quad \text{for } r \neq s$$

Repeat $KM^{-1}K\phi_s = \omega_s^2 K\phi_s \qquad \times$ by KM^{-1}

$$KM^{-1}KM^{-1}K\phi_s = \omega_s^2[KM^{-1}K]\phi_s \qquad \times \text{ by } \phi_r'$$

$$\underline{\underline{\phi_r'[KM^{-1}]^2 K\phi_s}} = \omega_s^2 \phi_r'[KM^{-1}K]\phi_s = \underline{\underline{0}} \qquad r \neq s$$

Repeat to obtain $\phi_r'[KM^{-1}]^h K\phi_s = 0$

<u>6-30</u>

$$K\phi_s = \omega_s^2 M\phi_s, \qquad M\underline{K^{-1}K}\phi_s = \omega_s^2 MK^{-1}M\phi_s$$

$$\therefore \; M\phi_s = \omega_s^2 MK^{-1}M\phi_s \qquad \text{ie } K^{-1}K = I$$

$$\phi_r' M\phi_s = \omega_s^2 \phi_r'[MK^{-1}M]\phi_s \underset{r \neq s}{=} 0 \qquad \text{ie } \phi_r' M\phi_s \underset{r \neq s}{=} 0$$

Repeat

$$MK^{-1}M\phi_s = \omega_s^2[MK^{-1}]^2 M\phi_s$$

$$\phi_r' MK^{-1}M\phi_s = \underline{\underline{\omega_s^2 \phi_r'[MK^{-1}]^2 M\phi_s}} = \underline{\underline{0}} \qquad r \neq s$$

Repeat h times.

6-31 Refer to Ex. 6.10-1 for ϕ_i & ω_i

For second mode

$$m_{22}\ddot{q}_2 + c_{22}\dot{q}_2 + k_{22}q_2 = -\ddot{u}_o(t)\sum_{i=1}^{10} m_i\phi_2(x_i)$$

$$m_{22} = \sum_{i=1}^{10} m_i\phi_2^2(x_i) = 5.5235\, m$$

$$c_{22} = 2\zeta_2\omega_2 m_{22} = 2\zeta_2\left(0.4451\sqrt{\frac{k}{m}}\right)m_{22} = 0.8902\zeta_2\sqrt{\frac{k}{m}}\, m_{22}$$

$$k_{22} = \omega_2^2 m_{22} = \left(0.1981\frac{k}{m}\right)m_{22}$$

$$\sum_{i=1}^{10} m_i\phi_2(x_i) = -2.2470\, m.$$

$$\therefore\quad \ddot{q}_2 + 0.8902\zeta_2\sqrt{\frac{k}{m}}\,\dot{q}_2 + 0.1981\frac{k}{m}q_2 = \frac{2.247}{5.5235}\ddot{u}_o(t)$$

$$= 0.4068\,\ddot{u}_o(t)$$

For 3rd mode

$$m_{33} = 8.5957\, m, \qquad c_{33}/m_{33} = 2\zeta_3\left(.7307\sqrt{\frac{k}{m}}\right) = 1.4614\zeta_2\sqrt{\frac{k}{m}}$$

$$k_{33} = 0.5339\frac{k}{m}, \qquad \sum_{i=1}^{10} m_i\phi_3(x_i) = 2.8095\, m$$

$$\ddot{q}_3 + 1.4614\zeta_3\sqrt{\frac{k}{m}}\,\dot{q}_3 + 0.5339\frac{k}{m}q_3 = -0.3268\,\ddot{u}_o(t)$$

6-32 From Ex. 6.10-1

$$\ddot{q}_1 + 0.2995\zeta_1\sqrt{\frac{k}{m}}\,\dot{q}_1 + 0.02235\frac{k}{m}q_1 = -1.2672\,\ddot{u}_o(t)$$

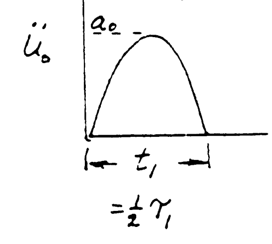
$$\omega_1^2 = 0.02235\frac{k}{m}, \qquad \omega_1 = 0.1495\sqrt{\frac{k}{m}} = \frac{2\pi}{\tau_1}$$

$$\therefore\ \tau_1 = \frac{2\pi}{.1495}\sqrt{\frac{m}{k}} = 42.028\sqrt{\frac{m}{k}}$$

6-32 Cont. From Ex. 6.10-1

$$\omega_2 = \frac{2\pi}{\tau_2} = 0.4451\sqrt{\frac{k}{m}}, \qquad \tau_2 = 14.1168\sqrt{\frac{m}{k}}$$

$$\omega_3 = 0.7307\sqrt{\frac{k}{m}}, \qquad \tau_3 = 8.5989\sqrt{\frac{m}{k}}$$



See FIG 4.5-5 for shock spectrum

$$\frac{t_1}{\tau_1} = 0.50 \qquad \left(\frac{xk}{F_0}\right)_{max} = 1.5$$

$$\frac{t_1}{\tau_1}\frac{\tau_1}{\tau_2} = \frac{t_1}{\tau_2} = .5\,\frac{42.028}{14.116} = 1.4886 \quad \therefore \left(\frac{xk}{F_0}\right)_{max} = 1.5$$

$$\frac{t_1}{\tau_3} = .50\,\frac{42.028}{8.5989} = 2.4438 \quad \therefore \left(\frac{xk}{F_0}\right)_{max} = 1.13$$

The right side of the DEs are:

$$-\ddot{u}_0(t)\frac{\sum m\phi_1}{\sum m\phi_1^2} = -1.2672\,\ddot{u}_0 = \frac{F_0}{m} \quad \text{by comparison}$$

$$\text{with } \ddot{q} + 2\zeta\omega_n\dot{q} + \omega_n^2 q = \frac{F_0}{m}$$

$$\therefore F_0 = -1.2672\, m\, a_0 \qquad \therefore \text{ in place of } \left(\frac{xk}{F_0}\right)_{max}$$

we use $\left(\frac{qk}{F_0}\right)_{max} = \frac{qk}{-1.2672\, m a_0}$ for mode 1 $= 1.5$

$$\therefore (q_1)_{max} = -1.5 \times 1.2672\,\frac{m}{k}a_0 = \underline{\underline{-1.9008\,\frac{m a_0}{k}}}$$

Similarly for 2nd mode & 3rd mode

$$(q_2)_{max} = 1.5 \times .4068\,\frac{m a_0}{k} = \underline{\underline{0.6102\,\frac{m a_0}{k}}}$$

$$(q_3)_{max} = 1.13 \times (-.3263)\,\frac{m a_0}{k} = \underline{\underline{-0.3693\,\frac{m a_0}{k}}}$$

6-32 Cont:

$$x(t) = \phi_1 q_1 + \phi_2 q_2 + \phi_3 q_3 + \cdots$$

but at 10^{th} floor $\phi_i = 1.0$

$$\therefore x(t) = q_1 + q_2 + q_3$$

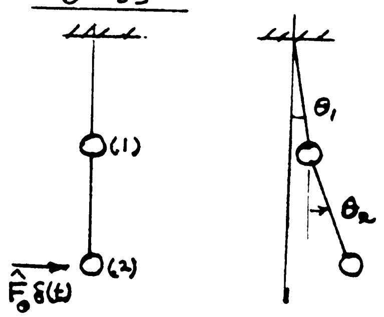
From Eq. (6.10-7)

$$|x(10)|_{max} = (q_1)_{max} + \sqrt{(q_2)^2_{max} + (q_3)^2_{max}}$$

$$= 1.90 + \sqrt{.610^2 + .369^2}$$

$$= 1.90 + 0.711 = \underline{\underline{2.61}} \ \frac{m a_0}{k}$$

6-33



Impulse = change in momentum

At $t=0$ mass at (2) will acquire a velocity of $\frac{\hat{F}_0}{m} = v(0) = \ell\dot{\theta}_2(0)$

$$\therefore \dot{\theta}_2(0) = \frac{\hat{F}}{m\ell} \qquad \dot{\theta}_1(0) = 0$$

$$\theta = \phi_1 q_1 + \phi_2 q_2$$

$$\therefore \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} .707 \\ 1.00 \end{Bmatrix} q_1 + \begin{Bmatrix} -.707 \\ 1.00 \end{Bmatrix} q_2$$

$$= \begin{Bmatrix} .707 \\ 1.00 \end{Bmatrix} A \sin .764\sqrt{\tfrac{g}{\ell}}\, t + \begin{Bmatrix} -.707 \\ 1.00 \end{Bmatrix} B \sin 1.85\sqrt{\tfrac{g}{\ell}}\, t$$

$$\sqrt{\frac{\ell}{g}} \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix} = .764 \begin{Bmatrix} .707 \\ 1.00 \end{Bmatrix} A \cos .764\sqrt{\tfrac{g}{\ell}}\, t + 1.85 \begin{Bmatrix} -.707 \\ 1.00 \end{Bmatrix} B \cos 1.85\sqrt{\tfrac{g}{\ell}}\, t$$

At $t = 0$

$$\sqrt{\frac{\ell}{g}} \begin{Bmatrix} 0 \\ \frac{\hat{F}}{m\ell} \end{Bmatrix} = .764 \begin{Bmatrix} .707 \\ 1.00 \end{Bmatrix} A + 1.85 \begin{Bmatrix} -.707 \\ 1.00 \end{Bmatrix} B$$

6-33 Cont

$$\therefore \quad 0 = (.764)\cancel{(.707)}A - (1.85)\cancel{(.707)}B$$

$$B = 0.413\,A$$

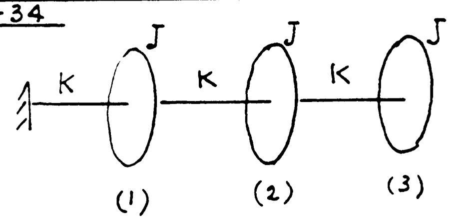
$$\sqrt{\frac{\ell}{g}}\,\frac{\hat{F}_o}{m\ell} = .764\,A + 1.85\,(.413\,A) = 1.528\,A$$

$$\therefore \quad A = 0.6544\sqrt{\frac{\ell}{g}}\,\frac{\hat{F}}{m\ell}$$

$$B = 0.2703\sqrt{\frac{\ell}{g}}\,\frac{\hat{F}}{m\ell}$$

$$\begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix} = \sqrt{\frac{\ell}{g}}\,\frac{\hat{F}}{m\ell}\left[(0.6544)\begin{Bmatrix}.707\\ 1.00\end{Bmatrix}\sin .764\sqrt{\frac{g}{\ell}}\,t + (.2703)\begin{Bmatrix}-.707\\ 1.00\end{Bmatrix}\sin 1.85\sqrt{\frac{g}{\ell}}\,t\right]$$

6-34



$$\lambda = \frac{J\omega^2}{K}$$

$$\phi = \begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{Bmatrix}$$

$$J\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{Bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\\ \ddot{\theta}_3\end{Bmatrix} + K\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&1\end{bmatrix}\begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{Bmatrix} = \begin{Bmatrix}0\\ 0\\ M_o u(t)\end{Bmatrix}$$

Let $\begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{Bmatrix} = P\begin{Bmatrix}q_1\\ q_2\\ q_3\end{Bmatrix}$ & premultiply by P'

$$P'JP\ddot{q} + P'KPq = P'\{M\}$$

$$m_{ii}\ddot{q}_i + k_{ii}q_i = M_o u(t)\,\theta_{3i}$$

$$\ddot{q}_i + \omega_i^2 q_i = \frac{M_o(\theta_3)_i}{m_{ii}}u(t)$$

$m_{11} = \phi_1' J\phi_1 = J(.328^2 + .591^2 + .737^2) = 1.00\,J$ $\qquad \omega_1^2 = .198\frac{k}{J}$

$m_{22} = J(.737^2 + .328^2 + .591^2) = 1.00\,J$ $\qquad \omega_2^2 = 1.555\frac{k}{J}$

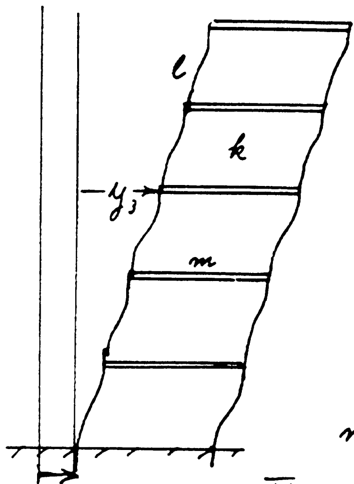
$m_{33} = 1.00\,J$ $\qquad \omega_3^2 = 3.247\frac{k}{J}$

Right side of eq $\frac{M_o(\theta_3)_i u(t)}{m_{ii}} =$

$\frac{.737}{J}M_o u(t)$ $\qquad -\frac{.591}{J}M_o u(t)$ $\qquad$ and $\qquad \frac{.328}{J}M_o u(t)$

6-35 Cont.

$$\frac{\Sigma m\phi_1}{\Sigma m\phi_1^2} = 2.097 \qquad \frac{\Sigma m\phi_2}{\Sigma m\phi_2^2} = 0.6602$$



For $K_R = \infty$, $\theta = 0$ and we have only translation of ground plus elastic translation of each floor. The more general case of $\theta \neq 0$ should be deferred until Ch 8, with Lagrange's Eq.

Write force eq. for one mass (say 3^{rd} floor)

$$m(\ddot{u} + \ddot{y}_3) = -k(y_3 - y_2) + k(y_4 - y_3)$$

The 5 equations like the above can be written in matrix form

$$m\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}\begin{Bmatrix}\ddot{y}_1\\\ddot{y}_2\\\ddot{y}_3\\\ddot{y}_4\\\ddot{y}_5\end{Bmatrix} + k\begin{bmatrix}2&-1&0&0&0\\-1&2&-1&0&0\\0&-1&2&-1&0\\0&0&-1&2&-1\\0&0&0&-1&1\end{bmatrix}\begin{Bmatrix}y_1\\y_2\\y_3\\y_4\\y_5\end{Bmatrix} = -m\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}\ddot{u}$$

or
$$M\ddot{y} + Ky = -M\ddot{u}$$

Let $y = \phi_1 q_1 + \phi_2 q_2$ & decouple by $P'KP$

or $\begin{Bmatrix}y_1\\y_2\end{Bmatrix} = \left[\{\phi_1\}\{\phi_2\}\right]\begin{Bmatrix}q_1\\q_2\end{Bmatrix} = Pq$

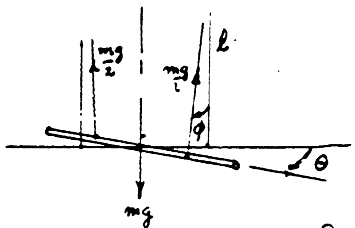
$$P'MP\ddot{q} + P'KPq = -P'M\ddot{u}$$

The modal eqs. become

$$\ddot{q}_1 + \omega_1^2 q_1 = -\frac{\Sigma m\phi_1}{\Sigma m\phi_1^2}\ddot{u}$$

$$\ddot{q}_2 + \omega_2^2 q_2 = -\frac{\Sigma m\phi_2}{\Sigma m\phi_2^2}\ddot{u}$$

6 - 36



$\phi \approx \frac{a\theta}{2\ell}$ Torsional Oscil.

$$J\ddot{\theta} = -\cancel{2}\frac{mg}{\cancel{2}}\frac{a\theta}{2\ell}\frac{a}{2} = -mg\frac{a^2}{4\ell}\theta$$

$$\therefore \omega_T^2 = \frac{mga^2}{4\ell J} = \frac{\cancel{mg}a^2}{4\ell\frac{\cancel{m}L^2}{12}} = 3\frac{g}{\ell}\left(\frac{a}{L}\right)^2$$

Out-of plane oscillation is that of simple pendulum with $\omega^2 = \frac{g}{\ell}$. For $\omega^2 = \omega_T^2$

$$\frac{g}{\ell} = 3\frac{g}{\ell}\left(\frac{a}{L}\right)^2 \qquad \therefore \frac{a}{L} = \frac{1}{\sqrt{3}}$$

With small accentricity to excite torsional oscil, there will be beating (see Prob. 1-4)

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6 - 37

Modal damping given as $\zeta_1 = .05$, $\zeta_2 = .13$

From Eq. (6.9-9)

$\omega_1 = .445\sqrt{\frac{k}{m}}$

$\omega_2 = 1.247\sqrt{\frac{k}{m}}$

$\omega_3 = 1.802\sqrt{\frac{k}{m}}$

$$2\zeta_1\omega_1 = \alpha + \beta\omega_1^2$$

$$2\zeta_2\omega_2 = \alpha + \beta\omega_2^2$$

$$2(\zeta_1\omega_1 - \zeta_2\omega_2) = \beta(\omega_1^2 - \omega_2^2)$$

$$\therefore \beta = \frac{2(\zeta_2\omega_2 - \zeta_1\omega_1)}{\omega_2^2 - \omega_1^2}, \qquad \alpha = \frac{2\omega_1\omega_2(\zeta_1\omega_2 - \zeta_2\omega_1)}{\omega_2^2 - \omega_1^2}$$

$$\therefore \beta = \frac{2(.05\times.445 - .13\times1.247)\sqrt{\frac{k}{m}}}{(.198 - 1.555)\frac{k}{m}} = \underline{\underline{0.2061\sqrt{\frac{m}{k}}}}$$

$$\alpha = \frac{2\times.445\times1.247(.05\times1.247 - .13\times.445)}{1.357}\sqrt{\frac{k}{m}} = \underline{\underline{0.00370\sqrt{\frac{k}{m}}}}$$

For 3rd mode

$$2\zeta_3\omega_3 = \alpha + \beta\omega_3^2 \qquad \zeta_3 = \frac{\alpha + \beta\omega_3^2}{2\omega_3}$$

$$\zeta_3 = \frac{.00370 + .2061\times3.247}{2\times1.802} = \underline{\underline{0.1867}}$$

6-38 $X_i' M X_j = (- \; - \; -)_i \, m [I] \begin{Bmatrix} - \\ - \\ - \end{Bmatrix}_i = m (- \; - \; -)_i \begin{Bmatrix} - \\ - \\ - \end{Bmatrix}_i$

With the modes given $X_i' M X_j = \begin{cases} 0 & \text{for } j \neq i \\ 1 & \text{for } j = i \end{cases}$

6-39 See Ex. 6.6-1 $\qquad C_i = \dfrac{\phi_i^T M u(0)}{\phi_i^T M \phi_i}$

From data given $\phi_i^T M \phi_i = 1.0\ m$

$$\therefore \; C_1 = \frac{(.737 \;\; .591 \;\; .328)\, \cancel{m}}{1.0\, \cancel{m}} \begin{Bmatrix} .520 \\ -.100 \\ .205 \end{Bmatrix} = .3914$$

$$C_2 = (-.591 \;\; .328 \;\; .737) \begin{Bmatrix} .520 \\ -.100 \\ .205 \end{Bmatrix} = -.1890 = -.4829\ C_1$$

$$C_3 = (.328 \;\; -.737 \;\; .591) \begin{Bmatrix} .520 \\ -.100 \\ .205 \end{Bmatrix} = .3654 = .9336\ C_1$$

6-40

$$X(t) = \sum_i X_i \left(A_i \sin \omega_i t + B_i \cos \omega_i t \right)$$

$$\dot{X}(t) = \sum_i \omega_i X_i \left(A_i \cos \omega_i t - B_i \sin \omega_i t \right)$$

$$\dot{X}(0) = \sum_i \omega_i X_i A_i$$

$$X_j' M \dot{X}(0) = \sum_i \omega_i X_j' M X_i A_i = \omega_j X_j' M X_j A_j$$

$$\therefore \; A_j = \frac{X_j' M \dot{X}(0)}{\omega_j X_j' M X_j} \qquad B_j = \frac{X_j' M X(0)}{X_j' M X_j}$$

6-41 The bending of the shaft is measured from the slope of the bearing, which is β. Thus y and θ in Example 6.1-4 must first be replaced by η and $(\theta-\beta)$ respectively.

$$\begin{Bmatrix} \eta \\ \theta-\beta \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} P \\ M \end{Bmatrix}$$

From geometry Fig P6-41

$$y = \frac{P}{k} + \beta\ell + \eta$$

$$M_B = K\beta = P\ell + M \qquad \therefore \beta = \frac{P\ell}{K} + \frac{M}{K}$$

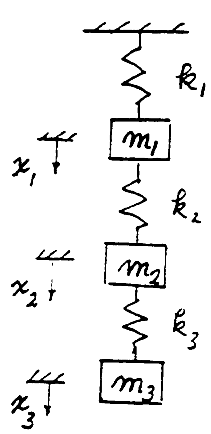
$$\text{and} \quad \eta = y - \frac{P}{k} - \beta\ell = y - \frac{P}{k} - \frac{P\ell^2}{K} - \frac{M\ell}{K}$$

$$\therefore \quad y - \frac{P}{k} - \frac{P\ell^2}{K} - \frac{M\ell}{K} = a_{11}P + a_{12}M$$

$$\theta - \frac{P\ell}{K} - \frac{M}{K} = a_{21}P + a_{22}M$$

$$\begin{Bmatrix} y \\ \theta \end{Bmatrix} = \begin{bmatrix} \left(a_{11} + \frac{1}{k} + \frac{\ell^2}{K}\right) & \left(a_{12} + \frac{\ell}{K}\right) \\ \left(a_{21} + \frac{\ell}{K}\right) & \left(a_{22} + \frac{1}{K}\right) \end{bmatrix} \begin{Bmatrix} P \\ M \end{Bmatrix} = \begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{bmatrix} \begin{Bmatrix} P \\ M \end{Bmatrix}$$

6-42



$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} (k_1+k_2) & -k_2 & 0 \\ -k_2 & (k_2+k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$M\ddot{X} + KX = 0$$

$$\left[-\omega^2 I + M^{-1}K\right]X = 0$$

$$\left[A - \lambda I\right]X = 0$$

where $A = M^{-1}K$

6-42 Cont.

$$M^{-1} = \begin{bmatrix} \frac{1}{m_1} & 0 & 0 \\ 0 & \frac{1}{m_2} & 0 \\ 0 & 0 & \frac{1}{m_3} \end{bmatrix}$$

$$A = M^{-1}K = \begin{bmatrix} \frac{1}{m_1} & 0 & 0 \\ 0 & \frac{1}{m_2} & 0 \\ 0 & 0 & \frac{1}{m_3} \end{bmatrix} \begin{bmatrix} (k_1+k_2) & -k_2 & 0 \\ -k_2 & (k_2+k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{m_1}(k_1+k_2) & -\frac{1}{m_1}k_2 & 0 \\ -\frac{1}{m_2}k_2 & \frac{1}{m_2}(k_2+k_3) & -\frac{1}{m_2}k_3 \\ 0 & -\frac{1}{m_3}k_3 & \frac{1}{m_3}k_3 \end{bmatrix} \quad \therefore \text{ not symmetric}$$

For symmetric matrix proceed as follows

$$Q = M^{\frac{1}{2}} = \begin{bmatrix} \sqrt{m_1} & 0 & 0 \\ 0 & \sqrt{m_2} & 0 \\ 0 & 0 & \sqrt{m_3} \end{bmatrix} \qquad Q^{-1} = \begin{bmatrix} \frac{1}{\sqrt{m_1}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{m_2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{m_3}} \end{bmatrix} = Q^{-T}$$

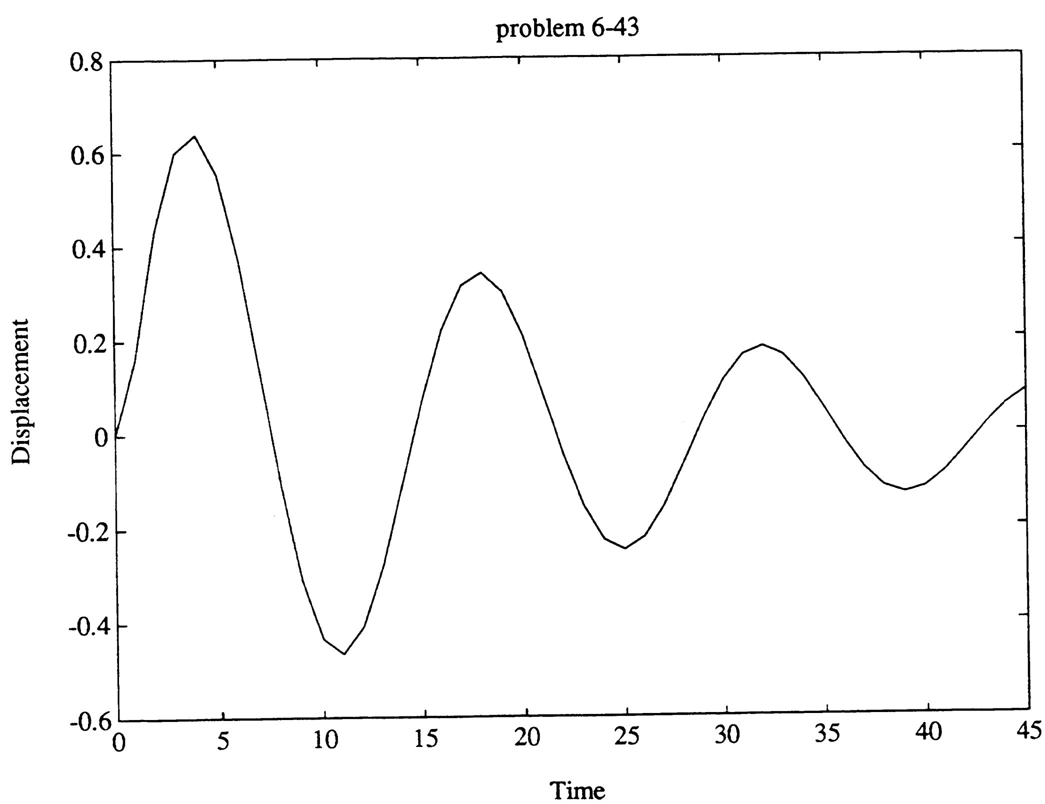
$$\text{New } A = Q^{-T}KQ^{-1} = \begin{bmatrix} \frac{1}{\sqrt{m_1}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{m_2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{m_3}} \end{bmatrix} \begin{bmatrix} (k_1+k_2) & -k_2 & 0 \\ -k_2 & (k_2+k_3) & -k_3 \\ 0 & -k_3 & k \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{m_1}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{m_2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{m_3}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{m_1}(k_1+k_2) & -\frac{1}{\sqrt{m_1 m_2}}k_2 & 0 \\ -\frac{1}{\sqrt{m_1 m_2}}k_2 & \frac{1}{m_2}(k_2+k_3) & -\frac{1}{\sqrt{m_2 m_3}}k_3 \\ 0 & -\frac{1}{\sqrt{m_2 m_3}}k_3 & \frac{1}{m_3}k_3 \end{bmatrix} \quad \therefore \text{ symmetric \& standard form.}$$

This procedure is valid for M a full symmetric matrix such as those encountered in the finite element formulation.

6-43

```
%This is the function file for problem 6-43
function [force]=f(t)
if t < 0.1
   force = 10*t;
elseif t < 0.3
force=1-(1.7/.2)*(t-.1);
elseif t<0.5
 force=-.7+(1.2/.2)*(t-.3);
elseif t<0.7
 force=.5-4*(t-.5);
elseif t<0.9
force=-.3+(5/2)*(t-.7);
elseif t<1
   force=.2-2*(t-.9);
else
force=0;
end
```



6-44

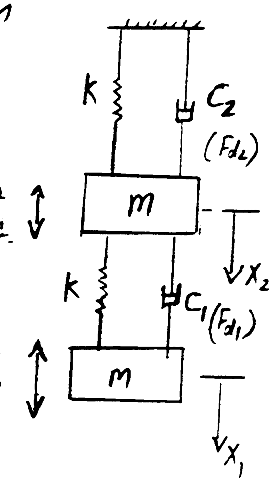
Let θ_1, θ_2 and θ_3 be the angular rotations of the three pendulums with the positive direction is the same for all of them. Assume small oscillations

Let $\theta_1 = 1$, $\theta_2 = \theta_3 = 0 \implies T_1 = k + mgl$, $T_2 = -k$, $T_3 = 0$
Let $\theta_1 = \theta_3 = 0$, $\theta_2 = 1 \implies T_1 = -k$, $T_2 = 2k + mgl$, $T_3 = -k$
Let $\theta_1 = \theta_2 = 0$, $\theta_3 = 1 \implies T_1 = 0$, $T_2 = -k$, $T_3 = k + mgl$

$$\therefore \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} k+mgl & -k & 0 \\ -k & 2k+mgl & -k \\ 0 & -k & k+mgl \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \quad \therefore [K] = \begin{bmatrix} k+mgl & -k & 0 \\ -k & 2k+mgl & -k \\ 0 & -k & k+mgl \end{bmatrix}$$

6-45

With linear dampers c_1 and c_2 the equation of motion is written as



$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c_1 & -c_1 \\ -c_1 & c_1+c_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

OR, $M\ddot{X} + C\ddot{X} + KX = F$

Let $X = \tilde{P}Y$ and post multiply the D.E. by $\tilde{P}^T$

$$\therefore \tilde{P}^T M \tilde{P} \ddot{Y} + \tilde{P}^T C \tilde{P} \dot{Y} + \tilde{P}^T K \tilde{P} Y = \tilde{P}^T F \quad \ldots\ldots (*)$$

$\tilde{P}^T M \tilde{P}$ and $\tilde{P}^T K \tilde{P}$ are diagonal matrices.

If $C = \alpha M + \beta K$, where α and β are constants (Rayleigh damping) then $\tilde{P}^T C \tilde{P}$ is diagonal and is equal to $\alpha I + \beta \Lambda$.

For example if $\underline{c_1 = c_2 = c}$ then $\alpha = 0$ and $\beta = c/k$.

In this case $(*)$ is written as

$$\ddot{Y}_i + (\alpha + \beta \omega_i^2)\dot{Y}_i + \omega_i^2 Y_i = \tilde{f}_i(t), \quad i = 1, 2$$

Now, if the damping force is proportional to the square of the velocity, we have

$$F_{d1} = \pm C(\dot{x}_1 - \dot{x}_2)^2 \quad \text{and} \quad F_{d2} = C\dot{x}_2^2$$

opposite to the direction of motion.

Transforming $\dot{x}_1$ and $\dot{x}_2$ into $\dot{y}_1$ and $\dot{y}_2$ via $\tilde{P}$ we express F_{d1} and F_{d2} in terms of $\dot{y}_1$ and $\dot{y}_2$.

<u>Answer</u>: Yes, it is possible to develop an equivalent damping approach to this problem.

Choose f_1 and f_2 such that $\tilde{f}_1(t)$ and $\tilde{f}_2(t)$ are periodic. Find the energy dissipated per cycle (in steady state) for $i=1,2$, i.e., W_{d1} and W_{d2}. By exciting the ith mode and letting the other one equal to zero plug in the steady state solution in the expressions for F_{d1} and F_{d2} in the y coordinates and find the dissipated energy per one cycle of the ith mode. (the sum of the energies dissipated by F_{d1} and F_{d2}), $i=1,2$. Now, equate W_{d1} and W_{d2} with the corresponding enery dissipation of quadratic damping and find

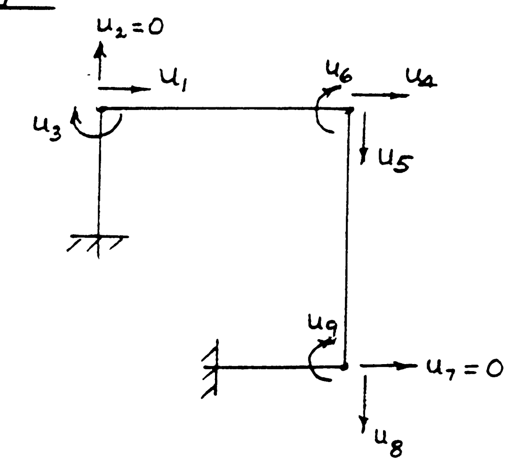
$$(\alpha + \beta \omega_i^2)_{eq} \quad , \quad i=1,2$$

$$\therefore \quad \tilde{P}^T C_{eq} \tilde{P} = \begin{bmatrix} (\alpha+\beta\omega_1^2)_{eq} & 0 \\ 0 & (\alpha+\beta\omega_2^2)_{eq} \end{bmatrix}$$

$\therefore$ C_{eq} can be found from this relation:

$$C_{eq} = \tilde{P}^{-T} \begin{bmatrix} (\alpha+\beta\omega_1^2)_{eq} & 0 \\ 0 & (\alpha+\beta\omega_2^2)_{eq} \end{bmatrix} \tilde{P}^{-1}$$

7-1



Disregard

$u_2 = u_7 = 0$

Constraints are

$u_1 = u_4$

$u_5 = u_8$

There are 7 coordinates and 2 constraint eqs.

$\therefore$ DOF $= 5$

7-2

7 Coordinates are $u_1 \quad u_3 \quad u_4 \quad u_5 \quad u_6 \quad u_8 \quad u_9$

4 of the above coordinates are associated with the two constraint equations, $u_1 \; u_4 \; u_5 \; u_8$.

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_4 \\ u_5 \\ u_8 \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix}$$

Above can be written as where u_4 & u_8 are chosen as gen. coords.

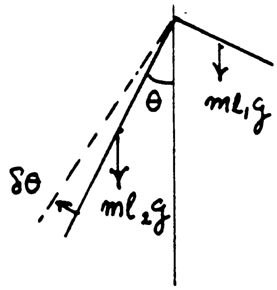
$$\begin{Bmatrix} u_1 \\ u_5 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} u_4 \\ u_8 \end{Bmatrix}$$

gen. coords.

$$\therefore \begin{Bmatrix} u_1 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_8 \\ u_9 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \\ u_6 \\ u_8 \\ u_9 \end{Bmatrix}$$

all coords.

7-3



$$-m\ell_2 g \frac{1}{2}\ell_2\left[\cos\theta - \cos(\theta+\delta\theta)\right] + m\ell_1 g \frac{\ell_1}{2}\left[\sin(\theta+\delta\theta) - \sin\theta\right] = 0$$

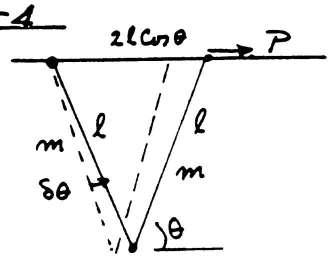
$$\ell_2^2\left[\cos\theta - (\cos\theta\cos\delta\theta - \sin\theta\sin\delta\theta\right] + \ell_1^2\left[\sin\theta\cos\delta\theta + \cos\theta\sin\delta\theta\right] = 0$$

$$\cos\delta\theta \cong 1, \quad \sin\delta\theta \cong \delta\theta$$

$$\ell_2^2\left[\sin\theta\cdot\delta\theta\right] + \ell_1^2\left[\cos\theta\cdot\delta\theta\right] = 0$$

$$\therefore \tan\theta = \left(\frac{\ell_1}{\ell_2}\right)^2$$

7-4



$$U = 2mg\frac{\ell}{2}\sin\theta$$

$$\delta U = mg\ell\cos\theta\,\delta\theta$$

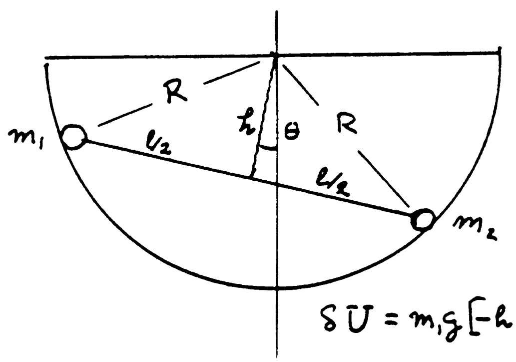
$$x = 2\ell\cos\theta$$

$$\delta x = 2\ell\sin\theta\,\delta\theta$$

$$\delta W = mg\ell\cos\theta\,\delta\theta - P2\ell\sin\theta\,\delta\theta = 0$$

$$\therefore \tan\theta = \frac{mg}{2P}$$

7-5



$$h = \sqrt{R^2 - \left(\frac{\ell}{2}\right)^2}$$

$$2h = \sqrt{(2R)^2 - \ell^2}$$

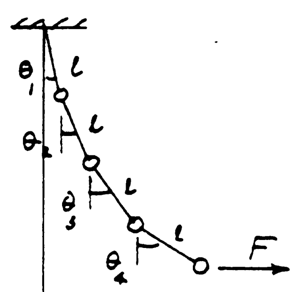
$$U = m_1 g\left[h\cos\theta - \frac{\ell}{2}\sin\theta\right] + m_2 g\left[h\cos\theta + \frac{\ell}{2}\sin\theta\right]$$

$$\delta U = m_1 g\left[-h\sin\theta\,\delta\theta - \frac{\ell}{2}\cos\theta\,\delta\theta\right] + m_2 g\left[-h\sin\theta\,\delta\theta + \frac{\ell}{2}\cos\theta\,\delta\theta\right] = 0$$

$$-(m_1+m_2)h\sin\theta + (m_2-m_1)\frac{\ell}{2}\cos\theta = 0$$

$$\tan\theta = \frac{(m_2-m_1)\ell/2}{(m_1+m_2)h} = \left(\frac{m_2-m_1}{m_2+m_1}\right)\frac{\ell}{\sqrt{(2R)^2+\ell^2}}$$

7-6



Top mass position $x = \ell \sin\theta_1$

$y = \ell \cos\theta_1$

Give θ_1 virtual displ $\delta\theta_1$

$$\delta x = \ell\cos\theta_1\,\delta\theta_1$$

$$\delta y = -\ell\sin\theta_1\,\delta\theta_1$$

$$\delta W = 4mg(-\ell\sin\theta_1\,\delta\theta_1) + F\ell\cos\theta_1\,\delta\theta_1 = 0$$

$$\therefore\ \tan\theta_1 = \frac{F}{4mg}$$

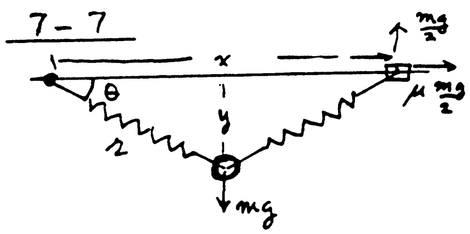
Give θ_2 virtual displ. $\delta\theta_2$

$$\delta W = 3mg(-\ell\sin\theta_2\,\delta\theta_2) + F\ell\cos\theta_2\,\delta\theta_2 = 0$$

$$\therefore\ \tan\theta_2 = \frac{F}{3mg}$$

etc. $\tan\theta_3 = \dfrac{F}{2mg}$, $\tan\theta_4 = \dfrac{F}{mg}$

7-7



$$y = r\sin\theta$$

$$\delta y = r\cos\theta\,\delta\theta$$

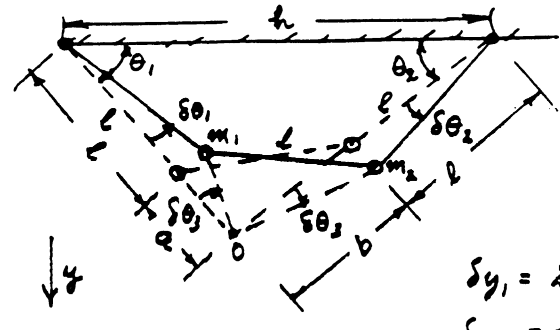
$$x = 2r\cos\theta$$

$$\delta x = -2r\sin\theta\,\delta\theta$$

$$\delta W = mg\,\delta y + \mu\frac{mg}{2}\,\delta x = 0 \qquad \therefore\ \tan\theta = \frac{1}{\mu}$$

7-8

dotted lines show original position in equilibrium



$$\delta W = m_1 g\,\delta y_1 - m_2 g\,\delta y_2 = 0$$

The middle length rotates $\delta\theta_3$ about O.

$$\ell\,\delta\theta_1 = a\,\delta\theta_3$$

$$\ell\,\delta\theta_2 = b\,\delta\theta_3$$

$$\delta y_1 = \ell\,\delta\theta_1\cos\theta_1 = a\,\delta\theta_3\cos\theta_1$$

$$\delta y_2 = \ell\,\delta\theta_2\cos\theta_2 = b\,\delta\theta_3\cos\theta_2$$

7-8 Cont.

$$\therefore \delta W = (m_1 g\, a\cos\theta_1 - m_2 g\, b\cos\theta_2)\delta\theta_3 = 0$$

$$\therefore \quad m_1 a\cos\theta_1 - m_2 b\cos\theta_2 = 0$$

Geometric Eqs.

$$(\ell+a)\cos\theta_1 + (\ell+b)\cos\theta_2 = h$$

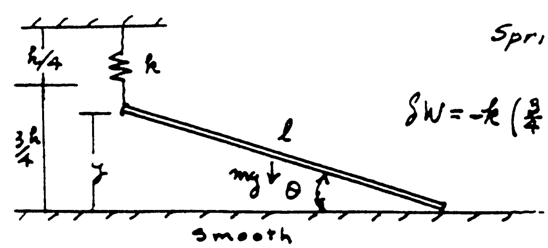
$$\ell\sin\theta_1 - \ell\sin\theta_2 = b\sin\theta_2 - a\sin\theta_1$$

$$\ell^2 = (a\cos\theta_1 + b\cos\theta_2)^2 + (b\sin\theta_2 - a\sin\theta_1)^2$$

4 eqs
4 unknowns
a, b, θ_1, θ_2

$$\sin(\theta_1+\theta_2) = \frac{h}{\ell}\,\frac{(m_1\cos\theta_1\sin\theta_2 - m_2\cos\theta_2\sin\theta_1)}{(m_1\cos\theta_1 - m_2\cos\theta_2)}$$

7-9



Stretch of spring $= \frac{3}{4}h - y$

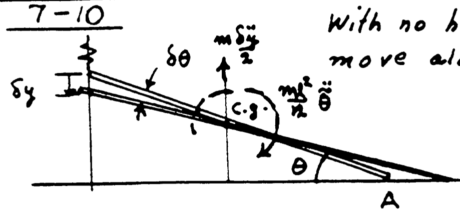
Spring force $= k(\frac{3}{4}h - y)$

$$\delta W = -k\left(\tfrac{3}{4}h - y\right)\delta y + mg\frac{\delta y}{2} = 0$$

$$\therefore \quad y = \frac{3}{4}h - \frac{mg}{2k}$$

$$\sin\theta = \frac{y}{\ell} = \frac{1}{\ell}\left(\frac{3h}{4} - \frac{mg}{2k}\right)$$

7-10



With no horizontal force on the bar, the c.g. must move along the vertical line.

Let $\delta\theta = \tilde\theta \quad \therefore \quad \theta = \theta_0 - \tilde\theta$

$$\sin(\theta_0 - \tilde\theta) \cong \sin\theta_0 - \tilde\theta\cos\theta_0$$

$$\cos(\theta_0 - \tilde\theta) \cong \cos\theta_0 + \tilde\theta\sin\theta_0$$

Forces are balanced at θ_0.

$$\delta y = \ell\sin\theta_0 - \ell\sin(\theta_0 - \tilde\theta) = (\ell\cos\theta_0)\tilde\theta$$

Spring force due to $\tilde\theta$ is $-k\delta y = -(k\ell\cos\theta_0)\tilde\theta$

$$\delta W = (-k\ell\cos\theta_0)\tilde\theta\cdot(\ell\cos\theta_0\cancel{\beta}) - \left(\frac{m}{2}\ell\cos\theta_0\ddot{\tilde\theta}\right)\left(\frac{\ell\cos\theta_0}{2}\cancel{\beta}\right) - \left(\frac{m\ell^2}{12}\ddot{\tilde\theta}\right)\cancel{\beta} = 0$$

$$\left(\frac{m\ell^2}{12} + \frac{m\ell^2}{4}\cos^2\theta_0\right)\ddot{\tilde\theta} + (k\ell^2\cos^2\theta_0)\tilde\theta = 0$$

For small θ_0 above eq. reduces to

$$\left(\frac{m\ell^2}{3}\right)\ddot{\tilde\theta} + (k\ell^2)\tilde\theta = 0$$

7-11

$$\left[(m\ell_1)\frac{\ell_1^2}{3} + (m\ell_2)\frac{\ell_2^2}{3}\right]\ddot{\theta} = -(m\ell_2 g)\frac{\ell_2}{2}\sin\theta + (m\ell_1 g)\frac{\ell_1}{2}\cos\theta$$

Let $\theta = \theta_0 + \theta_\sim$

$$\sin\theta \cong \sin\theta_0 + \theta_\sim \cos\theta_0$$

$$\cos\theta \cong \cos\theta_0 - \theta_\sim \sin\theta_0$$

but $-(m\ell_2 g)\frac{\ell_2}{2}\sin\theta_0 + (m\ell_1 g)\frac{\ell_1}{2}\cos\theta_0 = 0$

$$\therefore \left(\frac{\cancel{m}\ell_1^3}{3} + \frac{\cancel{m}\ell_2^3}{3}\right)\ddot{\theta}_\sim = -\frac{\cancel{m}g}{2}\left(\ell_2^2\cos\theta_0 + \ell_1^2\sin\theta_0\right)\theta_\sim$$

$$\ddot{\theta}_\sim + \frac{3}{2}g\left(\frac{\ell_2^2\cos\theta_0 + \ell_1^2\sin\theta_0}{\ell_1^3 + \ell_2^3}\right)\theta_\sim = 0$$

where $\tan\theta_0 = \left(\frac{\ell_1}{\ell_2}\right)^2$

7-12

$$T = \frac{1}{2}(m_1 + m_2)R^2\dot{\theta}^2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) = (m_1 + m_2)R^2\ddot{\theta} \qquad h = \sqrt{R^2 + \left(\frac{\ell}{2}\right)^2}$$

$$U = -(m_1 + m_2)gh\cos\theta - (m_2 - m_1)g\frac{\ell}{2}\sin\theta$$

$$\frac{\partial U}{\partial\theta} = (m_1 + m_2)gh\sin\theta - (m_2 - m_1)g\frac{\ell}{2}\cos\theta$$

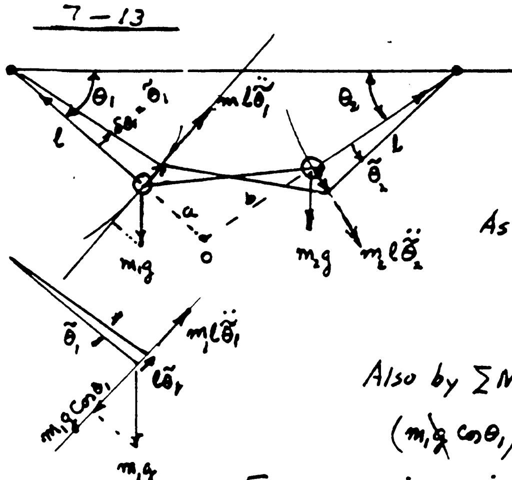
Lagranges Eq.

$$(m_1 + m_2)R^2\ddot{\theta} + (m_1 + m_2)gh\sin\theta - (m_2 - m_1)g\frac{\ell}{2}\cos\theta = 0$$

Let $\theta = \theta_0 + \theta_\sim$

$$(m_1 + m_2)R^2\ddot{\theta}_\sim + \left[(m_1 + m_2)gh\cos\theta_0 + (m_2 - m_1)g\frac{\ell}{2}\sin\theta_0\right]\theta_\sim = 0$$

7 — 13



Motion of each mass is along circle of radius l. Virtual displ $= l\tilde\theta_1$ and $l\tilde\theta_2$

where $\theta_1 = \theta_1 + \tilde\theta_1$ and $\theta_2 = \theta_2 + \tilde\theta_2$

As in Prob 7-8

$$\delta\theta_1 = \tilde\theta_1 = \frac{a}{l}\delta\theta_3 = \frac{a}{l}\tilde\theta_3$$

$$\delta\theta_2 = \tilde\theta_2 = \frac{b}{l}\delta\theta_3 = \frac{b}{l}\tilde\theta_3 = \frac{b}{a}\tilde\theta_1$$

Also by $\sum M_o$

$$(m_1 \cancel{g}\cos\theta_1)a = (m_2 \cancel{g}\cos\theta_2)b \qquad \text{same eq. as Prob 7-8}$$

For dynamics include inertia forces $m_1 l\ddot{\tilde\theta}_1$ & $m_2 l\ddot{\tilde\theta}_2$ in tangential direction $l\tilde\theta_1$ and $l\tilde\theta_2$. Then work done by virtual displ. $l\tilde\theta_1$ and $l\tilde\theta_2$ is

$$\delta W = (m_1 l\ddot{\tilde\theta}_1 - m_1 g\cos\theta_1)\cancel{l}\tilde\theta_1 + (m_2 l\ddot{\tilde\theta}_2 + m_2 g\cos\theta_2)\cancel{l}\tilde\theta_2 = 0$$

For small $\tilde\theta$ with $\theta = \bar\theta \pm \tilde\theta$

$$\cos\theta_1 = \cos\bar\theta_1 - \tilde\theta_1\sin\bar\theta_1$$

$$\cos\theta_2 = \cos\bar\theta_2 + \tilde\theta_2\sin\bar\theta_2$$

$$\tilde\theta_2 = \frac{b}{a}\tilde\theta_1 = \frac{m_1}{m_2}\frac{\cos\bar\theta_1}{\cos\bar\theta_2}\tilde\theta_1$$

$$+\left[m_1 l + m_2 l\left(\frac{b}{a}\right)^2\right]\ddot{\tilde\theta}_1\tilde\theta_1 - g\left\{m_1(\cancel{\cos\bar\theta_1} - \tilde\theta_1\sin\bar\theta_1)\tilde\theta_1 - m_2(\cancel{\cos\bar\theta_2} + \tfrac{b}{a}\tilde\theta_1\sin\bar\theta_2)\ \frac{b}{a}\tilde\theta_1\right\} = 0$$

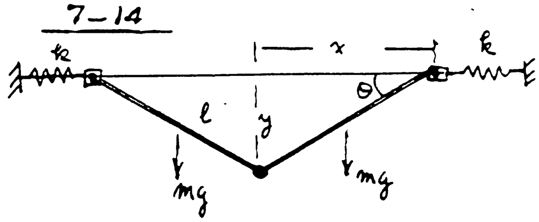
Since $m_1\cos\bar\theta_1\cdot a - m_2\cos\bar\theta_2\cdot b = 0$ from Prob. 8-6 & above

$$\left[m_1 l + m_2 l\left(\frac{b}{a}\right)^2\right]\ddot{\tilde\theta}_1\cancel{\tilde\theta_1} + g\left\{m_1\sin\bar\theta_1 + m_2\left(\frac{m_1\cos\bar\theta_1}{m_2\cos\bar\theta_2}\right)^2\right\}\tilde\theta_1\cancel{\tilde\theta_1} = 0$$

$$\left[m_1 + m_2\left(\frac{m_1\cos\bar\theta_1}{m_2\cos\bar\theta_2}\right)^2\right]\ddot{\tilde\theta}_1 + \frac{g}{l}\left[m_1\sin\bar\theta_1 + m_2\left(\frac{m_1}{m_2}\frac{\cos\bar\theta_1}{\cos\bar\theta_2}\right)^2\right]\tilde\theta_1 = 0$$

$$\therefore \omega_n^2 = \frac{g}{l}\cdot\frac{m_1\sin\bar\theta_1 + m_2\left(\frac{m_1\cos\bar\theta_1}{m_2\cos\bar\theta_2}\right)^2}{m_1 + m_2\left(\frac{m_1\cos\bar\theta_1}{m_2\cos\bar\theta_2}\right)^2}$$

7–14



Let θ_0 = equilib angle

Spring force = F_{s0} at $\theta = \theta_0$

$x = l\cos\theta \qquad \delta x = -l\sin\theta_0\,\delta\theta$

$y = l\sin\theta \qquad \delta y = l\cos\theta_0\,\delta\theta$

$$\delta W = 2mg\frac{\delta y}{2} + 2F_{s0}\,\delta x = 0$$

$$(mg\cos\theta_0 - 2F_{s0}\sin\theta_0)\,l\,\delta\theta = 0$$

If $F_s = 0$ at $\theta = 0$ then $F_{s0} = k(l - x) = kl(1-\cos\theta_0)$

$$\tan\theta_0 = \frac{mg}{2kl(1-\cos\theta_0)}$$ solve by trial for given value of $\frac{mg}{2kl}$

7–15

$$T = \frac{1}{2}m_0\left[(r\dot\theta)^2 + \dot r^2\right] + \frac{1}{2}\left(m\frac{l^2}{3}\right)\dot\theta^2$$

$$U = \frac{1}{2}k(r-r_0)^2 - m_0 g r\cos\theta - mg\frac{l}{2}\cos\theta$$

$$\frac{\partial T}{\partial\dot\theta} = m_0 r^2\dot\theta + m\frac{l^2}{3}\dot\theta \qquad \frac{\partial T}{\partial\theta} = 0$$

$$\frac{\partial U}{\partial\theta} = m_0 g r\sin\theta + mg\frac{l}{2}\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) - \frac{\partial T}{\partial\theta} + \frac{\partial U}{\partial\theta} = 0$$

$$m_0[r\ddot\theta + 2\dot r\dot\theta]r + \frac{ml^2}{3}\ddot\theta + \left(m_0 g r + mg\frac{l}{2}\right)\sin\theta = 0$$

$$\frac{\partial T}{\partial\dot r} = m_0\dot r \qquad \frac{\partial T}{\partial r} = m_0 r\dot\theta^2$$

$$\frac{\partial U}{\partial r} = -m_0 g\cos\theta + k(r-r_0)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot r}\right) - \frac{\partial T}{\partial r} + \frac{\partial U}{\partial r} = 0$$

$$m_0\ddot r - m_0 r\dot\theta^2 + k(r-r_0) - m_0 g\cos\theta = 0$$

7-16

$$T = \frac{1}{2}\int_0^\ell m\dot{y}^2\,dx$$

$$U = \frac{1}{2}k\,y^2(\ell) + \frac{1}{2}K\,y'^2(0) + \frac{1}{2}\int_0^\ell EI\left(\frac{d^2y}{dx^2}\right)^2 dx$$

$$y = \frac{x}{\ell}q_1 + q_2\sin\frac{\pi x}{\ell}$$

$$\dot{y}^2 = \left(\frac{x}{\ell}\right)^2\dot{q}_1^2 + 2\left(\frac{x}{\ell}\right)\dot{q}_1\dot{q}_2\sin\frac{\pi x}{\ell} + \dot{q}_2^2\sin^2\frac{\pi x}{\ell}$$

$$y' = \frac{1}{\ell}q_1 + q_2\frac{\pi}{\ell}\cos\frac{\pi x}{\ell}$$

$$y'^2 = \frac{1}{\ell^2}q_1^2 + \frac{2}{\ell}q_1q_2\frac{\pi}{\ell}\cos\frac{\pi x}{\ell} + q_2^2\left(\frac{\pi}{\ell}\right)^2\cos^2\frac{\pi x}{\ell}$$

$$y'' = -q_2\left(\frac{\pi}{\ell}\right)^2\sin\frac{\pi x}{\ell}$$

$$\frac{d}{dt}\frac{\partial T}{\partial\dot{q}_1} = \ddot{q}_1\int_0^\ell m\left(\frac{x}{\ell}\right)^2 dx + \ddot{q}_2\int_0^\ell \frac{x}{\ell}\sin\frac{\pi x}{\ell}\,dx = \frac{m\ell}{3}\ddot{q}_1 + \frac{\ell}{\pi}\ddot{q}_2$$

$$\frac{d}{dt}\frac{\partial T}{\partial\dot{q}_2} = \ddot{q}_1\int_0^\ell \frac{x}{\ell}\sin\frac{\pi x}{\ell}\,dx + \ddot{q}_2\int_0^\ell \sin^2\frac{\pi x}{\ell}\,dx = \frac{\ell}{\pi}\ddot{q}_1 + \frac{\ell}{2}\ddot{q}_2$$

$$\frac{\partial U}{\partial q_1} = kq_1 + \frac{K}{\ell^2}q_1 + \frac{K\pi}{\ell^2}q_2$$

$$\frac{\partial U}{\partial q_2} = \frac{K\pi}{\ell^2}q_1 + K\left(\frac{\pi}{\ell}\right)^2 q_2 + EI\left(\frac{\pi}{\ell}\right)^4\frac{\ell}{2}q_2$$

$$m\ell\begin{bmatrix}\frac{1}{3} & \frac{1}{2\pi}\\ \frac{1}{2\pi} & \frac{1}{2}\end{bmatrix}\begin{Bmatrix}\ddot{q}_1\\ \ddot{q}_2\end{Bmatrix} + \frac{EI}{\ell^3}\begin{bmatrix}\left(k+\frac{K}{\ell^2}\right)\frac{\ell^3}{EI} & -\frac{\pi K}{\ell^2}\frac{\ell^3}{EI}\\ -\frac{\pi K}{\ell^2}\frac{\ell^3}{EI} & \left(\frac{\pi^4}{2}+\frac{\pi^2 K}{\ell^2}\frac{\ell^3}{EI}\right)\end{bmatrix}\begin{Bmatrix}q_1\\ q_2\end{Bmatrix} = \{0\}$$

7-17

$$T = \frac{1}{2}\left(\frac{ml^2}{3}\right)\dot{\theta}_1^{\,2} + \frac{1}{2}\left(\frac{ml^2}{12}\right)\dot{\theta}_2^{\,2} + \frac{1}{2}m\left(l\dot{\theta}_1 + \frac{l}{2}\dot{\theta}_2\right)^2$$

$$U = \frac{1}{2}k\left(\frac{l}{2}\theta_1\right)^2 + mg\frac{l}{2}(1-\cos\theta_1) + mg\left[l(1-\cos\theta_1) + \frac{l}{2}(1-\cos\theta_2)\right]$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}_1} = \left(\frac{ml^2}{3}\right)\ddot{\theta}_1 + m\left(l\ddot{\theta}_1 + \frac{l}{2}\ddot{\theta}_2\right)l$$

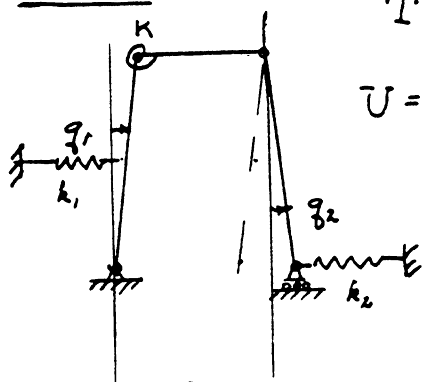
$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}_2} = \left(\frac{ml^2}{12}\right)\ddot{\theta}_2 + m\left(l\ddot{\theta}_1 + \frac{l}{2}\ddot{\theta}_2\right)\frac{l}{2}$$

$$\frac{\partial U}{\partial \theta_1} = k\left(\frac{l}{2}\theta_1\right) + mg\frac{l}{2}\sin\theta_1 + mg\,l\sin\theta_1$$

$$\frac{\partial U}{\partial \theta_2} = mg\frac{l}{2}\sin\theta_2$$

$$\begin{bmatrix} \left(\frac{ml^2}{3} + ml^2\right) & m\frac{l^2}{2} \\ m\frac{l^2}{2} & \left(\frac{ml^2}{12} + \frac{ml^2}{4}\right) \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} \left(k\frac{l}{2} + \frac{3}{2}mgl\right) & 0 \\ 0 & mg\frac{l}{2} \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \{0\}$$

7-18



$$T = \tfrac{1}{2}(m_1+m_2)(2\ell\dot{q}_1)^2 + \tfrac{1}{2}J_1\dot{q}_1^{\,2} + \tfrac{1}{2}J_2\dot{q}_2^{\,2}$$

$$U = \tfrac{1}{2}k_1(\ell q_1)^2 + \tfrac{1}{2}Kq_1^2 + \tfrac{1}{2}k_2(2\ell)^2(q_1+q_2)^2$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_1} = (m_1+m_2)(4\ell^2)\ddot{q}_1 + J_1\ddot{q}_1$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_2} = J_2\ddot{q}_2$$

$$\frac{\partial U}{\partial q_1} = \ell^2 k_1 q_1 + Kq_1 + k_2(4\ell^2)(q_1+q_2)$$

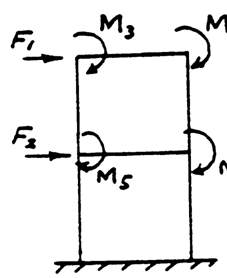
$$\frac{\partial U}{\partial q_2} = 4\ell^2 k_2(q_1+q_2)$$

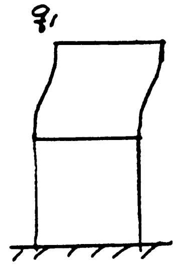
$$\left[(m_1+m_2)4\ell^2 + J_1\right]\ddot{q}_1 + \left[\ell^2 k_1 + K + 4\ell^2 k_2\right]q_1 + 4\ell^2 k_2 q_2 = 0$$

$$J_2\ddot{q}_2 + 4\ell^2 k_2(q_1+q_2) = 0$$

7-19

Refer to Fig 7.1-4. Assume $\ell_2 = \ell_1 = \ell$

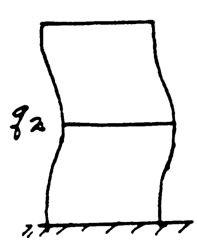


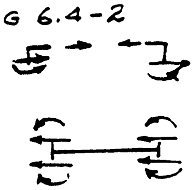


$$\begin{Bmatrix} F_1 \\ F_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{Bmatrix} = \frac{EI}{\ell^3}\begin{bmatrix} 24 & 0 & 0 & 0 & 0 & 0 \\ -24 & \vert & \vert & \vert & \vert & \vert \\ -6\ell & \vert & \vert & \vert & \vert & \vert \\ -6\ell & \vert & \vert & \vert & \vert & \vert \\ -6\ell & \vert & \vert & \vert & \vert & \vert \\ -6\ell & \vert & \vert & \vert & \vert & \vert \end{bmatrix}\begin{Bmatrix} q_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Examine FBD of each of the four corners for above

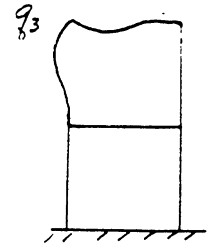
Refer to Fig 6.4-2





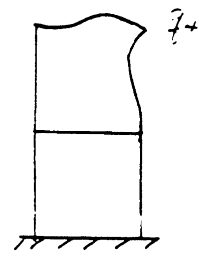
$$\begin{Bmatrix} F_1 \\ F_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{Bmatrix} = \frac{EI}{\ell^3}\begin{bmatrix} 0 & -24 & 0 & 0 & 0 & 0 \\ \vert & 48 & \vert & \vert & \vert & \vert \\ \vert & 6\ell & \vert & \vert & \vert & \vert \\ \vert & 6\ell & \vert & \vert & \vert & \vert \\ \vert & 0 & \vert & \vert & \vert & \vert \\ \vert & 0 & \vert & \vert & \vert & \vert \end{bmatrix}\begin{Bmatrix} 0 \\ q_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

7-19 Cont.

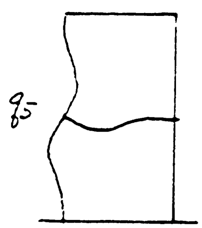


Examine FBD
of each corner

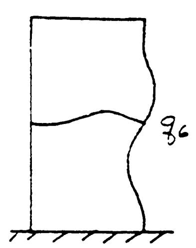
$$\begin{Bmatrix} F_1 \\ F_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{Bmatrix} = \frac{EI}{\ell^3}\begin{bmatrix} 0 & 0 & -6\ell & 0 & 0 & 0 \\ \cdot & \cdot & 6\ell & \cdot & \cdot & \cdot \\ \cdot & \cdot & 8\ell^2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 2\ell^2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 2\ell^2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot & \cdot \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ q_3 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$



$$\begin{Bmatrix} F_1 \\ F_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{Bmatrix} = \frac{EI}{\ell^3}\begin{bmatrix} 0 & 0 & 0 & -6\ell & 0 & 0 \\ \cdot & \cdot & \cdot & 6\ell & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2\ell^2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 8\ell^2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2\ell^2 & \cdot & \cdot \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ q_4 \\ 0 \\ 0 \end{Bmatrix}$$



$$\begin{Bmatrix} F_1 \\ F_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{Bmatrix} = \frac{EI}{\ell^3}\begin{bmatrix} 0 & 0 & 0 & 0 & -6\ell & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 2\ell^2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 12\ell^2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 12\ell^2 & \cdot \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ q_5 \\ 0 \end{Bmatrix}$$



$$\begin{Bmatrix} F_1 \\ F_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{Bmatrix} = \frac{EI}{\ell^3}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -6\ell \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 2\ell^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 2\ell^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 12\ell^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ q_6 \end{Bmatrix}$$

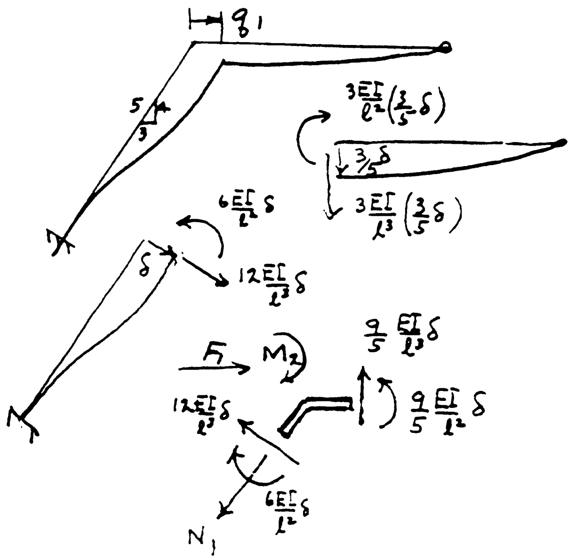
$$[k] = \frac{EI}{\ell^3}\left[\begin{array}{cc|cccc} 24 & -24 & -6\ell & -6\ell & -6\ell & -6\ell \\ -24 & 48 & 6\ell & 6\ell & 0 & 0 \\ \hline -6\ell & 6\ell & 8\ell^2 & 2\ell^2 & 2\ell^2 & 0 \\ -6\ell & 6\ell & 2\ell^2 & 8\ell^2 & 0 & 2\ell^2 \\ -6\ell & 0 & 2\ell^2 & 0 & 12\ell^2 & 2\ell^2 \\ -6\ell & 0 & 0 & 2\ell^2 & 2\ell^2 & 12\ell^2 \end{array}\right]$$

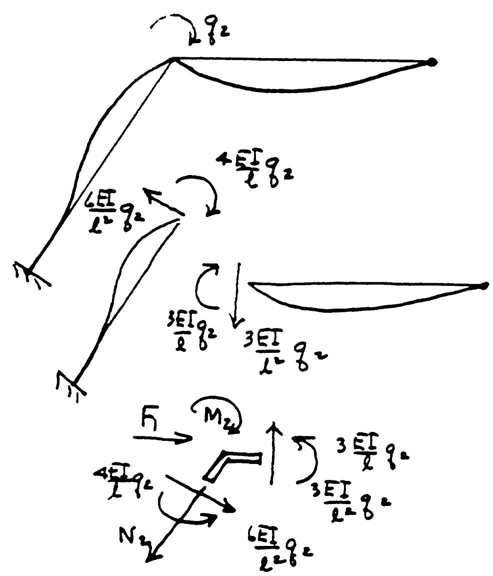
7-19 Cont.

$$[m] = \begin{bmatrix} 2m & 0 & 0 & 0 & 0 & 0 \\ 0 & 2m & 0 & 0 & 0 & 0 \\ 0 & 0 & J & 0 & 0 & 0 \\ 0 & 0 & 0 & J & 0 & 0 \\ 0 & 0 & 0 & 0 & J & 0 \\ 0 & 0 & 0 & 0 & 0 & J \end{bmatrix}$$

$$[m]\{\ddot{q}\} + [k]\{q\} = 0$$

7-20





$$\Sigma F_y = -\frac{4}{5}N_1 + \left(\frac{9}{5} + \frac{3}{5}\times 12\right)\frac{EI}{\ell^3}\delta = 0$$

$$\therefore N_1 = \frac{45}{4}\frac{EI}{\ell^3}\delta$$

$$\Sigma F_x = -F_1 + \left[\frac{3}{5}\left(\frac{45}{4}\right) + \frac{4}{5}\times 12\right]\frac{EI}{\ell^3}\delta = 0$$

$$F_1 = 16.35\frac{EI}{\ell^3}\delta$$

$$\Sigma M \qquad M_2 = \left(\frac{9}{5} - 6\right)\frac{EI}{\ell^2}\delta = -4.20\frac{EI}{\ell^2}\delta$$

but $\delta = \frac{5}{4}q_1$

$$\therefore F_1 = 20.44\frac{EI}{\ell^3}q_1$$

$$M_2 = -5.25\frac{EI}{\ell^2}q_1$$

$$\Sigma F_y = \frac{3EI}{\ell^2}q_2 - \frac{3}{5}\left(6\frac{EI}{\ell^2}\right)q_2 - \frac{4}{5}N_2 = 0$$

$$\therefore N_2 = -.75\frac{EI}{\ell^2}q_2$$

$\Sigma F_x = 0$ gives

$$F_1 = \left[\frac{3}{5}(-.75) - 6\times\frac{4}{5}\right]\frac{EI}{\ell^2}q_2$$

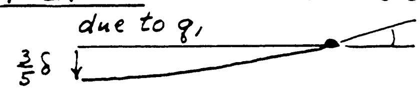
$$F_1 = -5.25\frac{EI}{\ell^2}q_2$$

$$M_2 = 7\frac{EI}{\ell}q_2$$

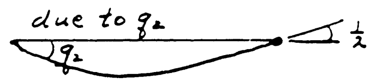
7-20 Cont.

$$\begin{Bmatrix}F_1\\ M_2\end{Bmatrix} = \frac{EI}{\ell^3}\begin{bmatrix}20.44 & -5.25\ell\\ -5.25\ell & 7.0\,\ell^2\end{bmatrix}\begin{Bmatrix}q_1\\ q_2\end{Bmatrix}$$

7-21 From FIG 6.4-2(a) rotation of right end is



$$\frac{3}{2}\left(\frac{3}{5}\frac{\delta}{\ell}\right) = \frac{3}{2}\left(\frac{3}{5}\times\frac{5}{4}\frac{q_1}{\ell}\right) = \frac{9}{8}\frac{q_1}{\ell}$$



$$\frac{1}{2}q_2$$

$$T = \frac{1}{2}(m_1+m_2)\dot{q}_1^{\,2} + \frac{1}{2}J_1\dot{q}_2^{\,2} + \frac{1}{2}J_2\left(\frac{9}{8}\frac{\dot{q}_1}{\ell} + \frac{1}{2}\dot{q}_2\right)^2$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_1} = (m_1+m_2)\ddot{q}_1 + J_2\left(\frac{9}{8\ell}\ddot{q}_1 + \frac{1}{2}\ddot{q}_2\right)\frac{9}{8\ell}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_2} = J_1\ddot{q}_2 + J_2\left(\frac{9}{8\ell}\ddot{q}_1 + \frac{1}{2}\ddot{q}_2\right)\frac{1}{2}$$

$$U = \frac{1}{2}(q_1\ q_2)\left[\ k\ \right]\begin{Bmatrix}q_1\\ q_2\end{Bmatrix} + \frac{1}{2}k_0 q_1^{\,2} + \frac{1}{2}K_0\left(\frac{9}{8\ell}q_1 + \frac{1}{2}q_2\right)^2$$

$$\frac{\partial U}{\partial q_{1,2}} = \begin{bmatrix}\left\{20.44\frac{EI}{\ell^3} + k_0 + \left(\frac{9}{8\ell}\right)^2 K_0\right\} & \left\{-5.25\frac{EI}{\ell^2} + \frac{1}{2}\times\frac{9}{8\ell}K_0\right\}\\ \left\{-5.25\frac{EI}{\ell^2} + \frac{1}{2}\times\frac{9}{8\ell}K_0\right\} & \left\{7.0\frac{EI}{\ell} + \frac{1}{4}K_0\right\}\end{bmatrix}\begin{Bmatrix}q_1\\ q_2\end{Bmatrix}$$

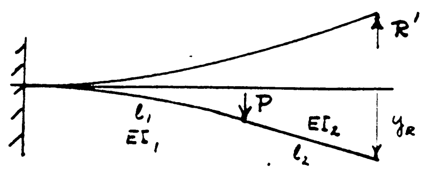
Eq. of motion

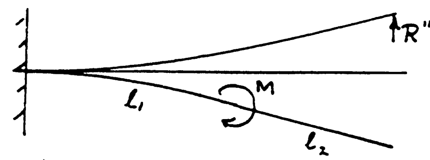
$$\begin{bmatrix}\left\{m_1+m_2 + \left(\frac{9}{8\ell}\right)^2 J_2\right\} & \left\{\frac{1}{2}\times\frac{9}{8\ell}J_2\right\}\\ \left\{\frac{1}{2}\times\frac{9}{8\ell}J_2\right\} & \left\{J_1 + \frac{1}{4}J_2\right\}\end{bmatrix}\begin{Bmatrix}\ddot{q}_1\\ \ddot{q}_2\end{Bmatrix}$$

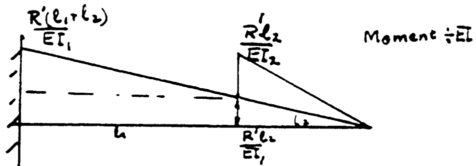
$$+\begin{bmatrix}\left\{20.44\frac{EI}{\ell^3} + k_0 + \left(\frac{9}{8\ell}\right)^2 K_0\right\} & \left\{-5.25\frac{EI}{\ell^2} + \frac{9}{16\ell}K_0\right\}\\ \left\{-5.25\frac{EI}{\ell^2} + \frac{9}{16\ell}K_0\right\} & \left\{7.0\frac{EI}{\ell} + \frac{1}{4}K_0\right\}\end{bmatrix}\begin{Bmatrix}q_1\\ q_2\end{Bmatrix} = \begin{Bmatrix}0\end{Bmatrix}$$

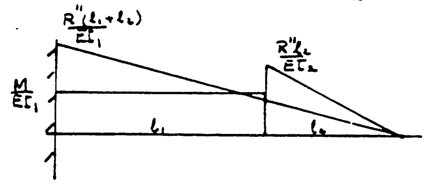
Assign numbers for m_1, m_2 J_1 J_2, k_0, K_0 & ℓ for normal mode determination.

7-22









Due to P

$$y_P = \frac{Pl_1^3}{3EI_1} \qquad y_P' = \frac{Pl_1^2}{2EI_1}$$

$$\therefore\ y_R = \frac{Pl_1^3}{3EI_1} + \left(\frac{Pl_1^2}{2EI_1}\right) l_2$$

Due to R'

$$y_R = \left(\frac{1}{2}\frac{R'l_2}{EI_2}\,l_2\right)\frac{2}{3}l_2 + \left(\frac{R'l_2}{EI_1}\cdot l_1\right)\left(l_2 + \frac{1}{2}l_1\right)$$

$$+\left(\frac{1}{2}\frac{R'l_1}{EI_1}\cdot l_1\right)\left(l_2 + \frac{2}{3}l_1\right)$$

Equate y_R due to P and due to R' (use $\frac{EI_1}{EI_2}=2$)

$$\frac{P}{\cancel{EI_1}}\left[\frac{l_1^3}{3} + \frac{l_1^2 l_2}{2}\right] = \frac{R'}{\cancel{EI_1}}\left[\frac{2}{3}l_2^3 + l_1 l_2^2 + l_2 l_1^2 + \frac{l_1^3}{3}\right]$$

$$= R'[b]$$

Due to M

$$y_M = \left(\frac{M}{EI_1}\cdot l_1\right)\frac{l_1}{2}$$

$$y_M' = \frac{M}{EI_1}l_1$$

$$y_R = \frac{M}{EI_1}\frac{l_1^2}{2} + \frac{Ml_1 l_2}{EI_1}$$

Due to R''

y_R same as that for R' with R' replaced by R''

Equate y_R due to M and due to R''

$$\frac{M}{\cancel{EI_1}}\left(\frac{l_1^2}{2} + l_1 l_2\right) = \frac{R''}{\cancel{EI_1}}\left[\frac{2}{3}l_2^3 + l_1 l_2^2 + l_2 l_1^2 + \frac{l_1^3}{3}\right]$$

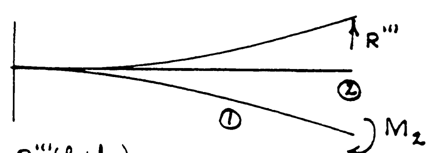
$$= R''[b]$$

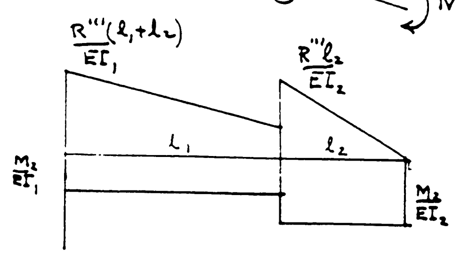
$$M_1 = -M - Pl_1 + R(l_1+l_2) \qquad \text{where } R = R' + R''$$

$$R = \frac{P\left(\frac{1}{3}l_1^3 + \frac{1}{2}l_1^2 l_2\right) + M\left(\frac{1}{2}l_1^2 + l_1 l_2\right)}{\frac{2}{3}l_2^3 + l_1 l_2^2 + l_2 l_1^2 + \frac{1}{3}l_1^3}$$

7-23

Need stiffness matrix for M, P and M_2 at end of beam





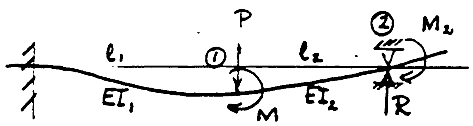
$$y_R = \frac{M_2}{EI_1}l_1\left(l_2 + \frac{l_1}{2}\right) + \frac{M_2}{EI_2}\frac{l_2^2}{2}$$

$$y_R = \frac{R'''}{EI_1}\left[\frac{2}{3}l_2^3 + l_1 l_2^2 + l_2 l_1^2 + \frac{1}{3}l_1^3\right]$$

Equate

$$\frac{M_2}{\cancel{EI_1}}\left[l_1 l_2 + \frac{l_1^2}{2} + 2\cdot\frac{l_2^2}{2}\right] = \frac{R'''}{\cancel{EI_1}}[b]$$

7-23 Cont:



Determine y_1 θ_1 y_2 θ_2

Set. $y_2 = 0$ to find R

Determine flexibility matrix & invert.

$EI_1 = 2\ EI_2$

$$y_1 = \frac{P\ell_1^3}{3EI_1} + \frac{M\ell_1^2}{2EI_1} + \frac{M_2\ell_1^2}{2EI_1} - \frac{1}{2}\left[\frac{R(\ell_1+\ell_2)\ell_1}{EI_1} - \frac{R\ell_1\ell_2}{EI_1}\right]\frac{2}{3}\ell_1 - \frac{R\ell_2}{EI_1}\frac{\ell_1^2}{2}$$

$$\theta_1 = \frac{P\ell_1^2}{2EI_1} + \frac{M\ell_1}{EI_1} + \frac{M_2\ell_1}{EI_1} - \frac{1}{2}\left[\frac{R(\ell_1+\ell_2)}{EI_1} + \frac{R\ell_2}{EI_1}\right]\ell_1$$

$$y_2 = \frac{1}{2}\frac{P\ell_1^2}{EI_1}\left(\frac{2}{3}\ell_1+\ell_2\right) + \frac{M\ell_1}{EI_1}\left(\frac{1}{2}\ell_1+\ell_2\right) + \frac{M_2\ell_1}{EI_1}\left(\frac{\ell_1}{2}+\ell_2\right) + \frac{M_2}{EI_2}\frac{\ell_2^2}{2}$$
$$- \frac{1}{2}\frac{R(\ell_1+\ell_2)}{EI_1}(\ell_1+\ell_2)\frac{2}{3}(\ell_1+\ell_2) - \frac{1}{2}\left[\frac{R\ell_2^2}{EI_2} - \frac{R\ell_2\ell_2}{EI_1}\right]\frac{2}{3}\ell_2$$

$$\theta_2 = \frac{P\ell_1^2}{2EI_1} + \frac{M\ell_1}{EI_1} + \frac{M_2\ell_1}{EI_1} + \frac{M_2\ell_2}{EI_2} - \frac{1}{2}\frac{R(\ell_1+\ell_2)^2}{EI_1} - \frac{1}{2}\left(\frac{R\ell_2}{EI_2} - \frac{R\ell_2}{EI_1}\right)\ell_2$$

To simplify algebra let $\ell_1 = \ell_2 = \ell$ & $EI_1 = 2EI_2$, result is

$$\begin{Bmatrix} y_1 \\ \theta_1 \\ \theta_2 \\ y_2 = 0 \end{Bmatrix} = \frac{1}{EI_1}\begin{bmatrix} \frac{\ell^3}{3} & \frac{\ell^2}{2} & \frac{\ell^2}{2} & -\frac{5}{6}\ell^3 \\ \frac{\ell^2}{2} & \ell & \ell & -\frac{3}{2}\ell^2 \\ \frac{\ell^2}{2} & \ell & 3\ell & -\frac{5}{2}\ell^2 \\ \frac{5}{6}\ell^3 & \frac{3}{2}\ell^2 & \frac{5}{2}\ell^2 & -3\ell^3 \end{bmatrix}\begin{Bmatrix} P \\ M \\ M_2 \\ R \end{Bmatrix}$$

from $y_2 = 0$ $\quad R = \frac{5}{18}P + \frac{1}{2}\frac{M}{\ell} + \frac{5}{6}\frac{M_2}{\ell}$ $\quad$ Subst. into above to obtain

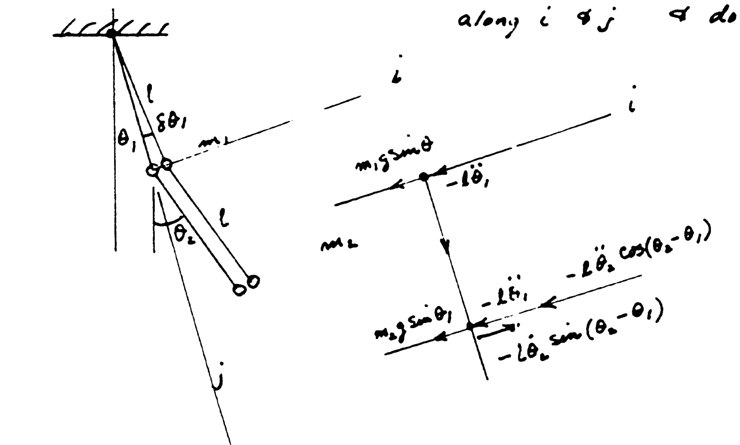
$$\begin{Bmatrix} y_1 \\ \theta_1 \\ \theta_2 \end{Bmatrix} = \frac{1}{EI_1}\begin{bmatrix} 0.1018\ell^3 & 0.0833\ell^2 & -0.1944\ell^2 \\ 0.0833\ell^2 & 0.25\ell & -0.25\ell \\ -0.1944\ell^2 & -0.25\ell & 0.0916\ell \end{bmatrix}\begin{Bmatrix} P \\ M \\ M_2 \end{Bmatrix}$$

invert by computer program to obtain stiffness matrix

Let numeric value of $\ell = 1.0$

7-24

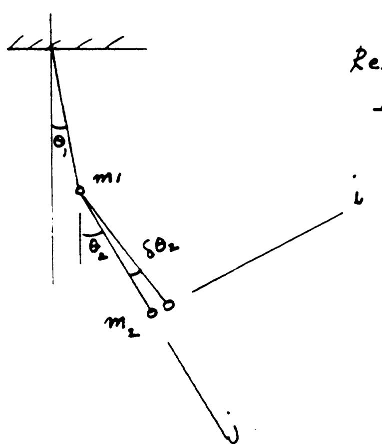
Resolve all forces including inertia forces along i & j & dot product with $l\delta\theta_1 i$



$$\Sigma(F - m\ddot r)\cdot l\delta\theta_1 i = -(m_1+m_2) g \sin\theta_1 - (m_1+m_2) l\ddot\theta_1 - m_2 l\ddot\theta_2 \cos(\theta_2-\theta_1) + m_2 l\dot\theta_2^2 \sin(\theta_2-\theta_1) = 0$$

or $$(m_1+m_2) l\ddot\theta_1 + m_2 l\ddot\theta_2 \cos(\theta_2-\theta_1) - m_2 l\dot\theta_2 \sin(\theta_2-\theta_1) + (m_1+m_2) g \sin\theta_1 = 0$$

Resolve all forces along new i j $\perp$ & // to l_2 and dot with $l\delta\theta_2 i$



$$m_2 l\ddot\theta_2 + m_2 l\ddot\theta_1 \cos(\theta_2-\theta_1) + m_2 l\dot\theta_1^2 \sin(\theta_2-\theta_1) + m_2 g \sin\theta_2 = 0$$

Try checking these equations from Lagrange's eqs.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta_i}\right) - \frac{\partial T}{\partial\theta_i} + \frac{\partial U}{\partial\theta_i} = 0$$

7-25

$$T = \frac{1}{2}(3m)\dot{q}_1^2 + \frac{1}{2} m \dot{q}_2^2$$

$$U = \frac{1}{2} k q_1^2 + \frac{1}{2} k (q_2 - q_1)^2 + \frac{1}{2} k q_2^2$$

$$L = T - U = \frac{1}{2}(3m)\dot{q}_1^2 + \frac{1}{2} m \dot{q}_2^2 - \frac{1}{2} k q_1^2 - \frac{1}{2} k (q_2 - q_1)^2 - \frac{1}{2} k q_2^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) = 3m \ddot{q}_1 , \quad \frac{\partial L}{\partial q_1} = -k q_1 + k (q_2 - q_1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) = m \ddot{q}_2 , \quad \frac{\partial L}{\partial q_2} = -k (q_2 - q_1) - k q_2$$

Eq. in matrix form

$$m \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + k \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

7-26

Assume that the table can move only in the vertical direction and the mass m is rotating with angular speed ω around the vertical axis. Let r be the distance between m and the axis of rotation. Denote the vertical displacement by x

Kinetic energy $T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m (r\omega)^2 = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m r^2 \omega^2$

Potential energy $U = \frac{1}{2}(4k)x^2 = 2kx^2$

Let θ be the angular displacement of m. Therefore, $\omega = d\theta/dt = \dot{\theta}$.

We use Lagrange's equations to find the differential equations of motion.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial U}{\partial x} = 0 \Longrightarrow m\ddot{x} + 4kx = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial U}{\partial \theta} = 0 \Longrightarrow \ddot{\theta} = 0$$

In matrix form

$$\begin{bmatrix} m & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 4k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

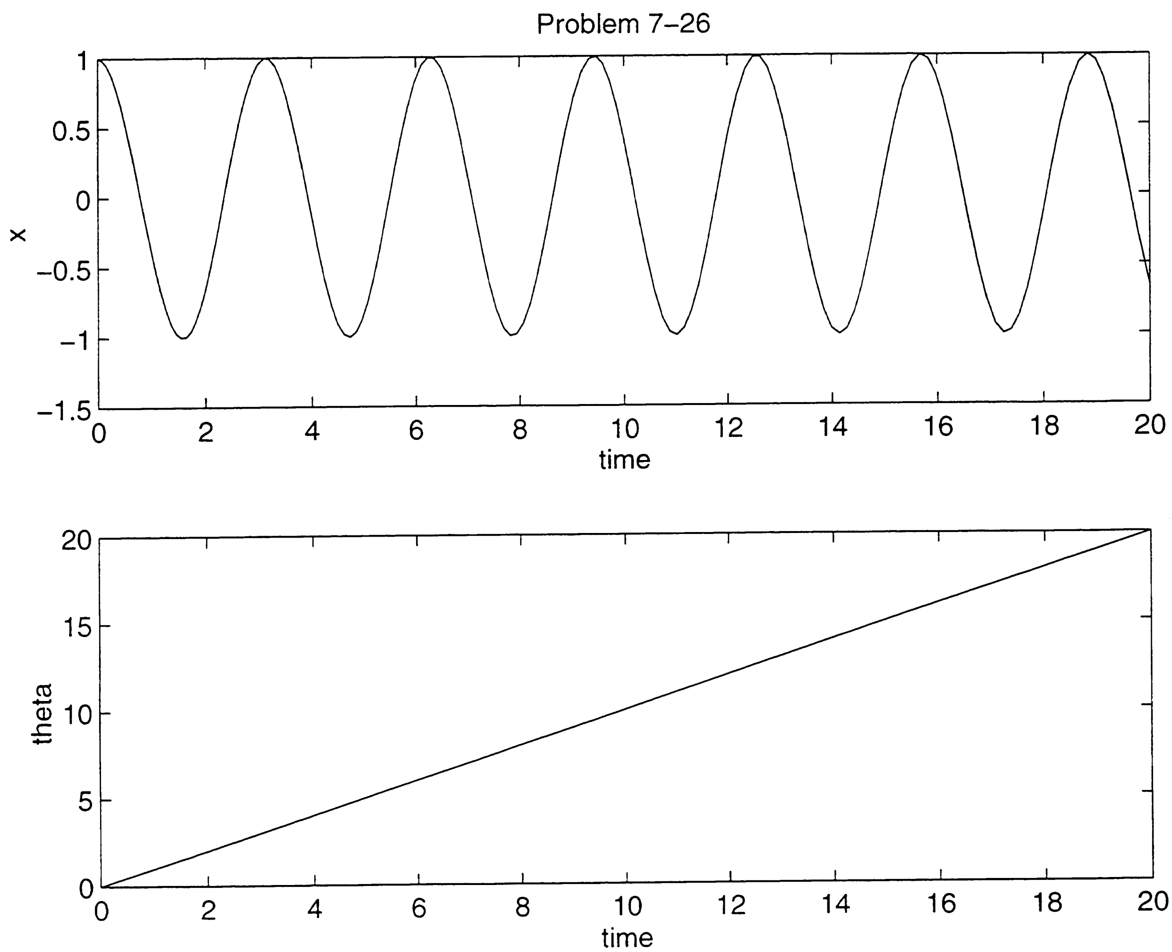
7-26 cont

```
clear
'Problem 7-26 Solution'
global m k
m=input('Enter the value of the mass m (kg) : ')
k=input('Enter the value of the spring constant k (N/m) : ')
x0=input('Enter the value of the four dimensional initial condition [x xd theta thetad] :
')
tf=input('Enter the value of the final time : ')
t0=0;
[t,x]=ode45('p726d',t0,tf,x0);
subplot(2,1,1);plot(t,x(:,1));xlabel('time');ylabel('x');title('Problem 7-26')
subplot(2,1,2);plot(t,x(:,3));xlabel('time');ylabel('theta');
```

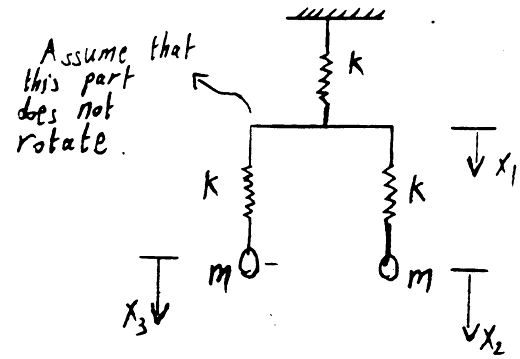
```
function xdot=p726d(t,x)
global m k
xdot=[x(2);
      -4*(k/m)*x(1);
    x(4);
    0];
```

7-26 cont.

$m = k = 1, \quad X_o = [1 \quad 0 \quad 0 \quad 1]$



7-27



Kinetic energy $T = \frac{1}{2} m \dot{x}_2^2 + \frac{1}{2} m \dot{x}_3^2$

Potential energy $U = \frac{1}{2} k x_1^2 + \frac{1}{2} k (x_3 - x_1)^2 + \frac{1}{2} k (x_2 - x_1)^2$

Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) - \frac{\partial T}{\partial x_1} + \frac{\partial U}{\partial x_1} = 0 \Longrightarrow k x_1 - k(x_3 - x_1) - k(x_2 - x_1) = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) - \frac{\partial T}{\partial x_2} + \frac{\partial U}{\partial x_2} = 0 \Longrightarrow m\ddot{x}_2 + k(x_2 - x_1) = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_3}\right) - \frac{\partial T}{\partial x_3} + \frac{\partial U}{\partial x_3} = 0 \Longrightarrow m\ddot{x}_3 + k(x_3 - x_1) = 0$$

In matrix form

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 3k & -k & -k \\ -k & k & 0 \\ -k & 0 & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since x_1 can be expressed in terms of x_2 and x_3 we write $\left(x_1 = \frac{1}{3}(x_2 + x_3)\right)$

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \frac{k}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

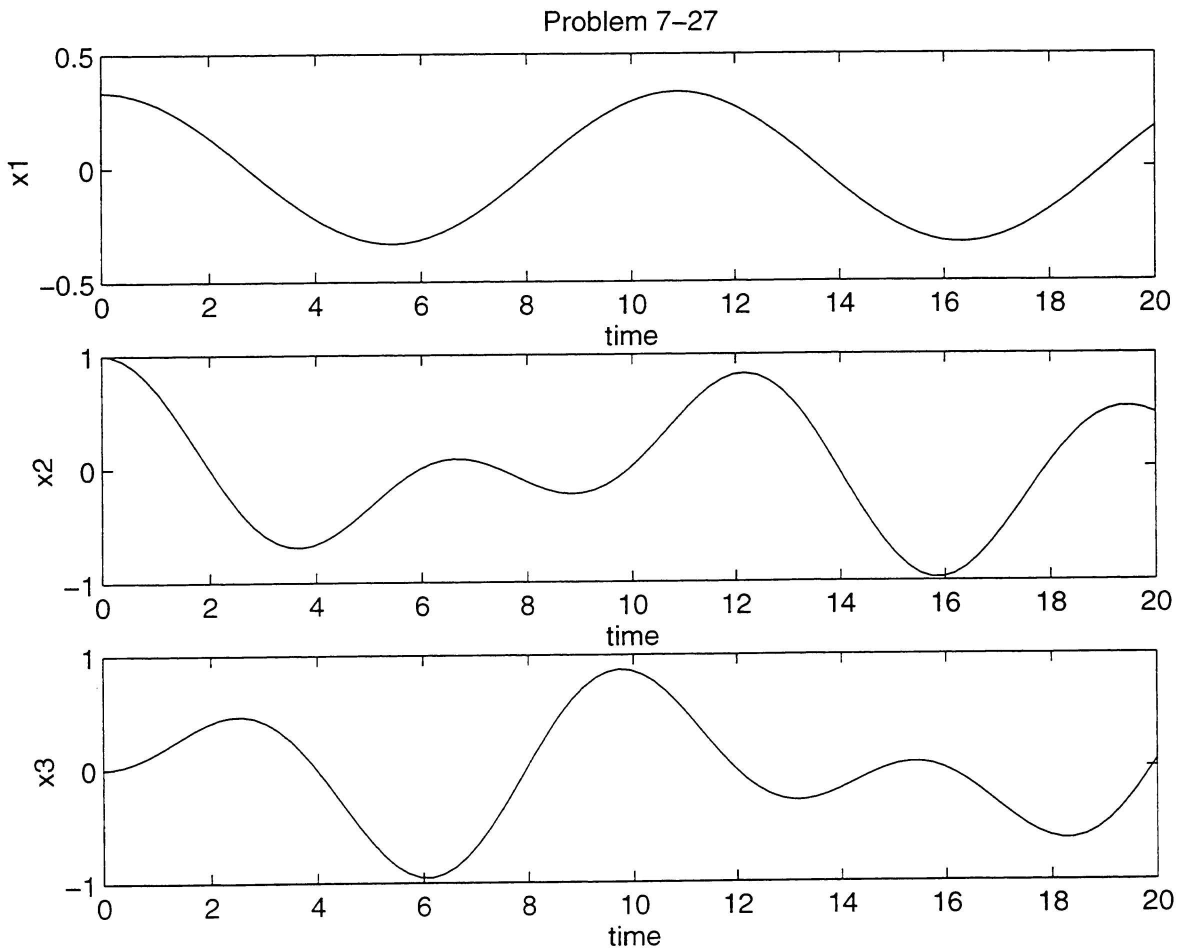
7-27 cont

```
clear
'Problem 7-27 Solution'
global m k
m=input('Enter the value of the mass m (kg) : ')
k=input('Enter the value of the spring constant k (N/m) : ')
x0=input('Enter the value of the four dimensional initial condition [x2 x2d x3 x3d] : ')
tf=input('Enter the value of the final time : ')
t0=0;
[t,x]=ode45('p727d',t0,tf,x0);
z=(1/3)*(x(:,1)+x(:,3));
subplot(3,1,1);plot(t,z);xlabel('time');ylabel('x1');title('Problem 7-27')
subplot(3,1,2);plot(t,x(:,1));xlabel('time');ylabel('x2');
subplot(3,1,3);plot(t,x(:,3));xlabel('time');ylabel('x3');
```

```
function xdot=p727d(t,x)
global m k
xdot=[x(2);
      -(k/(3*m))*(2*x(1)-x(3));
    x(4);
      -(k/(3*m))*(2*x(3)-x(1))];
```

7-27 cont

$m = k = 1$, $X_0 = [1 \quad 0 \quad 0 \quad 0]$



8-1

Eq. in terms of stiffness

$$[-\lambda M + K]X = 0$$

when premultiplied by $K^{-1} = [a]$

$$[[a][M] - \bar{\lambda} I]X = 0 \qquad \text{where } \bar{\lambda} = \frac{k}{\omega^2 m}$$

Characteristic Eq.

$$\left| \begin{bmatrix} .5 & .5 & .5 \\ .5 & 1.5 & 1.5 \\ .5 & 1.5 & 2.5 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \bar{\lambda} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} (1-\bar{\lambda}) & .5 & .5 \\ 1 & (1.5-\bar{\lambda}) & 1.5 \\ 1 & 1.5 & (2.5-\bar{\lambda}) \end{vmatrix} = 0$$

Using first column as pivot

$$(1-\bar{\lambda}) \begin{vmatrix} (1.5-\bar{\lambda}) & 1.5 \\ 1.5 & (2.5-\bar{\lambda}) \end{vmatrix} - \begin{vmatrix} .5 & .5 \\ 1.5 & (2.5-\bar{\lambda}) \end{vmatrix} + \begin{vmatrix} .5 & .5 \\ (1.5-\bar{\lambda}) & 1.5 \end{vmatrix} = 0$$

$$(1-\bar{\lambda})(1.5-\bar{\lambda})(2.5-\bar{\lambda}) - 2.25(1-\bar{\lambda}) - (1.25 - .5\bar{\lambda}) + .75 + .75 - .75 + .5\bar{\lambda} = 0$$

$$-\bar{\lambda}^3 + 5\bar{\lambda}^2 - 4.50\bar{\lambda} + 1 = 0$$

8-2 Convert the matrix problem to the standard eigenvalue problem. In matlab

```
»M=[ 2 0 0 ; 0 1 0 ; 0 0 1]
```

8-2 cont.

```
M =

     2     0     0
     0     1     0
     0     0     1

»K=[ 3 -1 0 ; -1 2 -1; 0 -1 1 ]

K =

     3    -1     0
    -1     2    -1
     0    -1     1

»t=inv(M)*K

t =

    1.5000   -0.5000         0
   -1.0000    2.0000   -1.0000
         0   -1.0000    1.0000

»[V,D]=eig (t)

V =
      φ1         φ2         φ3
    0.7569    0.3031    0.2333
    0.2189   -0.8422    0.5808
   -0.6158    0.4458    0.7799
```

These vectors can be normalized to give

$$\phi_1 = \begin{pmatrix} -1.2291 \\ -.3555 \\ 1.000 \end{pmatrix} \quad \phi_2 = \begin{pmatrix} .6799 \\ -1.8892 \\ 1.000 \end{pmatrix}$$

$$\phi_3 = \begin{pmatrix} .2991 \\ .7447 \\ 1.000 \end{pmatrix}$$

```
D =

    1.3554         0         0
         0    2.8892         0
         0         0    0.2554
```

»omega=sqrt(D) $= \omega\sqrt{\frac{m}{K}}$

```
omega =

    1.1642         0         0
         0    1.6998         0
         0         0    0.5053
```

8-3 Characteristic equation in terms of stiffness

$$\begin{bmatrix} (3-2\lambda_i) & -1 & 0 \\ -1 & (2-\lambda_i) & -1 \\ 0 & -1 & (1-\lambda_i) \end{bmatrix} = 0 \qquad \lambda = \frac{\omega^2 m}{k}$$

For mode 2 substitute $\lambda_2 = 1.3554$

$$\begin{bmatrix} .2892 & -1 & 0 \\ -1 & .6446 & -1 \\ 0 & -1 & -.3554 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$\div$ 1^{st} row by .2892 and add to 2^{nd} row

$$\begin{bmatrix} 1 & -3.4578 & 0 \\ 0 & -2.8132 & -1 \\ 0 & -1 & -.3554 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ 1.0 \end{Bmatrix} = 0$$

From 3^{rd} row obtain $x_2 = -.3554$

" 1^{st} row $x_1 - .3554(-3.4578) = 0 \quad \therefore x_1 = -1.2289$

$$\therefore \phi_2 = \begin{Bmatrix} -1.2289 \\ -.3554 \\ 1.000 \end{Bmatrix}$$

For ϕ_3 do same to obtain $\phi_3 = \begin{Bmatrix} .6800 \\ -1.889 \\ 1.000 \end{Bmatrix}$

See Prob 8-2 computer results.

8-4

$$\begin{bmatrix} (3-2\lambda) & -1 & 0 \\ -1 & (2-\lambda) & -1 \\ 0 & -1 & (1-\lambda) \end{bmatrix} = 0$$

For method of cofactors it is not necessary to alter the first row

$C_{11} = \begin{vmatrix} (2-\lambda) & -1 \\ -1 & (1-\lambda) \end{vmatrix}$ let $\lambda_1 = .2554$ then $C_{11} = .2991$

$C_{12} = -\begin{vmatrix} -1 & -1 \\ 0 & (1-\lambda) \end{vmatrix}$ $\qquad C_{12} = .7447$

8-4 Cont.

$$C_{13} = \begin{vmatrix} -1 & (2-\lambda) \\ 0 & -1 \end{vmatrix} = 1.0 \qquad \therefore \phi_1 = \begin{Bmatrix} .2991 \\ .7447 \\ 1.00 \end{Bmatrix}$$

For ϕ_2 sub $\lambda_2 = 1.3554$ etc. into same cofactors
can also start with flexability equation

8-5

Eq of motion in terms of cofactors of a column is not proportional to the eigen vectors.

8-6

$$\left| -\left(\frac{\omega^2 m}{k}\right)\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \right| = \begin{vmatrix} (2-\lambda) & -1 & 0 \\ -1 & (2-\lambda) & -1 \\ 0 & -1 & (1-\lambda) \end{vmatrix} = 0 \qquad \lambda = \frac{\omega^2 m}{k}$$

Gaus Elim. $\lambda_1 = 0.198$

$$\begin{bmatrix} 1.802 & -1 & 0 \\ -1 & 1.199 & -1 \\ 0 & -1 & .802 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ 1.0 \end{Bmatrix} = 0 \qquad x_2 = .802 \qquad x_1 = \frac{.802}{1.802} = .445$$

$$\therefore \phi_1 = \begin{Bmatrix} .445 \\ .802 \\ 1.000 \end{Bmatrix}$$

when there are zeros in matrix Gaus Elim. method is not necessary for a 3x3 problem

$\lambda_2 = 1.555$

$$\begin{bmatrix} .445 & -1 & 0 \\ -1 & .445 & -1 \\ 0 & -1 & -.555 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ 1.0 \end{Bmatrix} = 0 \qquad \begin{aligned} x_3 &= 1.0 \\ x_2 &= -.555 \\ x_1 &= -1.247 \end{aligned} \qquad \phi_2 = \begin{Bmatrix} -1.247 \\ -.555 \\ 1.000 \end{Bmatrix}$$

$\lambda_3 = 3.247$

$$\begin{bmatrix} -1.247 & -1 & 0 \\ -1 & -1.247 & -1 \\ 0 & -1 & -2.247 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ 1.0 \end{Bmatrix} = 0 \qquad \begin{aligned} x_3 &= 1.0 \\ x_2 &= -2.247 \\ x_1 &= 1.802 \end{aligned} \qquad \phi_3 = \begin{Bmatrix} 1.802 \\ -2.247 \\ 1.000 \end{Bmatrix}$$

8-6 cont.

```
»D=[ 2 -1 0 ; -1 2 -1 ;0 -1 1]

D =

     2    -1     0
    -1     2    -1
     0    -1     1

»[V,D]=eig(D)

V =
```

$$\begin{array}{ccc} \phi_1 & \phi_2 & \phi_3 \\ 0.7370 & -0.3280 & 0.5910 \\ 0.3280 & -0.5910 & -0.7370 \\ -0.5910 & -0.7370 & 0.3280 \end{array}$$

These can be Normalized to get

$$\phi_1 = \begin{pmatrix} -1.2470 \\ -.5548 \\ 1.000 \end{pmatrix} \quad \phi_2 = \begin{pmatrix} .4450 \\ .8019 \\ 1.000 \end{pmatrix}$$

$$\phi_3 = \begin{pmatrix} -1.8018 \\ -2.247 \\ 1.000 \end{pmatrix}$$

```
D =

    1.5550         0         0
         0    0.1981         0
         0         0    3.2470
```

»sqrt(D) $= \omega \sqrt{\frac{K}{m}}$

```
ans =

    1.2470         0         0
         0    0.4450         0
         0         0    1.8019

»
```

8 – 7

From Example 6.1-1, the flexibility matrix is

$$\frac{1}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$ and the equation of motion is

$$\left[\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} - \left(\frac{k}{\omega^2 m}\right) I \right]\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = 0 \qquad \text{Let } \bar{\lambda} = \frac{k}{\omega^2 m}$$

characteristic equation

$$\begin{vmatrix} (1-\bar{\lambda}) & 1 & 1 \\ 1 & (2-\bar{\lambda}) & 2 \\ 1 & 2 & (3-\bar{\lambda}) \end{vmatrix} = 0 \qquad \begin{aligned} \bar{\lambda}_1 &= 5.0505 \\ \bar{\lambda}_2 &= 0.6431 \\ \bar{\lambda}_3 &= 0.3080 \end{aligned}$$

Subst. $\bar{\lambda} = 5.0505$

$$\begin{vmatrix} -4.0505 & 1 & 1 \\ 1 & -3.0505 & 2 \\ 1 & 2 & -2.0505 \end{vmatrix} \div 1^{st}\text{row by } 4.0505 \begin{vmatrix} -1 & .2469 & .2469 \\ 1 & -3.0505 & 2 \\ 1 & 2 & -2.0505 \end{vmatrix}$$

add rows 1 and 2 to form new row 2

$$\begin{vmatrix} -4.0505 & 1 & 1 \\ 0 & -2.8036 & 2.2469 \\ 1 & 2 & -2.0505 \end{vmatrix}$$

next add rows 1 & 3 to form new row 3

$$\begin{vmatrix} - & - & - \\ - & - & - \\ 0 & 2.2469 & -1.8036 \end{vmatrix}$$

result is

$$\begin{vmatrix} -4.0505 & 1 & 1 \\ 0 & -2.8036 & 2.2469 \\ 0 & 2.2469 & -1.8036 \end{vmatrix}$$

continue to reduce term 3,2 to zero

$$\begin{bmatrix} -4.0505 & 1 & 1 \\ 0 & -2.8036 & 2.2469 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ 1.0 \end{Bmatrix} = 0$$

$$x_2 = \frac{2.2469}{2.8036} = .8014$$

$$x_1 = \frac{1.8014}{4.0505} = .4447 \qquad \phi_1 = \begin{Bmatrix} .4447 \\ .8014 \\ 1.000 \end{Bmatrix}$$

See Prob 8-6 for ϕ_2 & ϕ_3

repeat with $\bar{\lambda}_2 = 0.6431$ and $\bar{\lambda}_3 = .3080$ $\quad \phi_2 = \begin{Bmatrix} -1.247 \\ -.5548 \\ 1.000 \end{Bmatrix} \quad \phi_3 = \begin{Bmatrix} 1.818 \\ -2.247 \\ 1.000 \end{Bmatrix}$

8-8 Change k_i and m_i as per assigned by instructor

8-9 With $k_i = k$ and $m_i = m$, the equation of motion is

$$\left\{ \lambda \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} + \begin{bmatrix} 3 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \right\} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \qquad \lambda = \frac{\omega^2 m}{k}$$

characteristic equation

$$\begin{vmatrix} (3-\lambda) & -1 & -1 & 0 \\ -1 & (2-\lambda) & -1 & 0 \\ -1 & -1 & (3-\lambda) & -1 \\ 0 & 0 & -1 & (1-\lambda) \end{vmatrix} = (3-\lambda)\begin{vmatrix} (2-\lambda) & -1 & 0 \\ -1 & (3-\lambda) & -1 \\ 0 & -1 & (1-\lambda) \end{vmatrix} + \begin{vmatrix} -1 & -1 & 0 \\ -1 & (3-\lambda) & -1 \\ 0 & -1 & (1-\lambda) \end{vmatrix} - \begin{vmatrix} -1 & (2-\lambda) & 0 \\ -1 & -1 & -1 \\ 0 & 0 & (1-\lambda) \end{vmatrix} = 0$$

$$= (3-\lambda)(2-\lambda)\begin{vmatrix} (3-\lambda) & -1 \\ -1 & (1-\lambda) \end{vmatrix} + (3-\lambda)\begin{vmatrix} -1 & -1 \\ 0 & (1-\lambda) \end{vmatrix} - \begin{vmatrix} (3-\lambda) & -1 \\ -1 & (1-\lambda) \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ 0 & (1-\lambda) \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ 0 & (1-\lambda) \end{vmatrix}$$

$$+ (2-\lambda)\begin{vmatrix} -1 & -1 \\ 0 & (1-\lambda) \end{vmatrix} = 0$$

$$= \left[(3-\lambda)(2-\lambda)-1\right]\left[(3-\lambda)(1-\lambda)-1\right] - \left[(7-2\lambda)(1-\lambda)\right] = 0$$

$$\therefore \lambda^4 - 9\lambda^3 + 25\lambda^2 - 21\lambda + 3 = 0$$

The eigenvalues can either be determined from the roots of the characteristic equation or directly using the command eig.

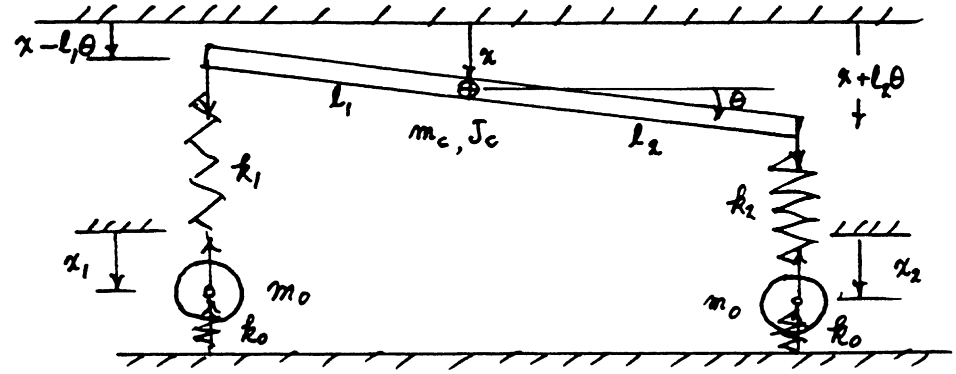
```
»c=[1 -9 25 -21 3]

c =

     1    -9    25   -21     3

»roots(c)

ans =

    4.1796
    3.4882
    1.1538
    0.1783
```

```
»D=[ 3 -1 -1 0 ; -1 2 -1 0 ; -1 -1 3 -1 ; 0 0 -1 1]

D =

     3    -1    -1     0
    -1     2    -1     0
    -1    -1     3    -1
     0     0    -1     1

»[V,D]=eig(D)

V =

    0.6371   -0.5800    0.3679   -0.3497
   -0.6679   -0.0883    0.5672   -0.4737
    0.3569    0.7725    0.1120   -0.5131
   -0.1434   -0.2430   -0.7283   -0.6245


D =

    3.4882         0         0         0
         0    4.1796         0         0
         0         0    1.1538         0
         0         0         0    0.1783

»
```

8 - 10

Optional (see computer programs for Prob. 8-9)

8 - 11 4 DOF Auto



$$m_c\ddot{x}_c = k_1[x_1-(x-l_1\theta)] + k_2[x_2-(x+l_2\theta)]$$

$$J_c\ddot{\theta} = -k_1l_1[x_1-(x-l_1\theta)] + k_2l_2[x_2-(x+l_2\theta)]$$

$$m_0\ddot{x}_1 = -k_0x_1 - k_1[x_1-(x-l_1\theta)]$$

$$m_0\ddot{x}_2 = -k_0x_2 - k_2[x_2-(x+l_2\theta)]$$

$$\begin{bmatrix} m_c & & & \\ & J_c & & \\ & & m_0 & \\ & & & m_0 \end{bmatrix}\begin{Bmatrix}\ddot{x}\\ \ddot{\theta}\\ \ddot{x}_1\\ \ddot{x}_2\end{Bmatrix} + \begin{bmatrix} (k_1+k_2) & (k_2l_2-k_1l_1) & -k_1 & -k_2 \\ (k_2l_2-k_1l_1) & (k_1l_1^2+k_2l_2^2) & k_1l_1 & -k_2l_2 \\ -k_1 & k_1l_1 & (k_1+k_0) & 0 \\ -k_2 & -k_2l_2 & 0 & (k_2+k_0)\end{bmatrix}\begin{Bmatrix}x\\ \theta\\ x_1\\ x_2\end{Bmatrix} = \begin{Bmatrix}0\\0\\0\\0\end{Bmatrix}$$

8 - 12 4 DOF Auto

$$m = \left[\begin{array}{cc|cc} 100 & & & \\ & 1600 & & \\ \hline & & 4.969 & \\ & & & 4.969 \end{array}\right] \qquad k = \left[\begin{array}{cc|cc} 5000 & 3500 & -2400 & -2600 \\ 3500 & 127{,}250 & 10{,}800 & -14300 \\ \hline -2400 & 10{,}800 & 40{,}800 & 0 \\ -2600 & -14300 & 0 & 41000 \end{array}\right]$$

8-12 cont.

```
»M=[ 100 0 0 0 ; 0 1600 0 0 ; 0 0 4.969 0 ; 0 0 0 4.969]

M =

   1.0e+03 *

    0.1000         0         0         0
         0    1.6000         0         0
         0         0    0.0050         0
         0         0         0    0.0050

»K=[ 5000 3500 -2400 -2600; 3500 127250 10800 -14300;
-2400 10800 40800 0 ; -2600 -14300 0 41000]

K =

        5000        3500       -2400       -2600
        3500      127250       10800      -14300
       -2400       10800       40800           0
       -2600      -14300           0       41000

»D=inv(M)*K

D =

   1.0e+03 *

    0.0500    0.0350   -0.0240   -0.0260
    0.0022    0.0795    0.0067   -0.0089
   -0.4830    2.1735    8.2109         0
   -0.5232   -2.8778         0    8.2512

»[v,d]=eig(D)

v =

    0.9940   -0.0031    0.0030   -0.6996
   -0.0671   -0.0011   -0.0008   -0.6473
    0.0767   -0.0202   -0.9998    0.1314
    0.0398    0.9998   -0.0203   -0.2727
```

rotation | wheel vibration | translation

8-12 Cont.

d =

1.0e+03 *

rotation	wheel vibration		translation
0.0447	0	0	0
0	8.2560	0	0
0	0	8.2141	0
0	0	0	0.0768

8-13

$$m = \begin{bmatrix}500 & 0\\ 0 & 100\end{bmatrix} \qquad [a] = \frac{\ell^3}{48EI}\begin{bmatrix}1 & -1.5\\ -1.5 & 6\end{bmatrix}$$

$$[a]^{-1} = [k] = \frac{48EI}{\ell^3}\cdot\frac{1}{3.75}\begin{bmatrix}6 & 1.5\\ 1.5 & 1\end{bmatrix} = \frac{12.80EI}{\ell^3}\begin{bmatrix}6 & 1.5\\ 1.5 & 1\end{bmatrix}$$

Let $\bar{\lambda} = \left(\frac{12.8EI}{\omega^2 \ell^3}\right)$ and

$$\left|\begin{bmatrix}500 & \\ & 100\end{bmatrix} - \bar{\lambda}\begin{bmatrix}6 & 1.5\\ 1.5 & 1\end{bmatrix}\right| = 0$$

```
»M=[500 0 ; 0 100]

M =

   500     0
     0   100

»K=[ 6 1.5 ; 1.5 1]

K =

    6.0000    1.5000
    1.5000    1.0000

»d=inv(K)*M

d =

  133.3333  -40.0000
 -200.0000  160.0000
```

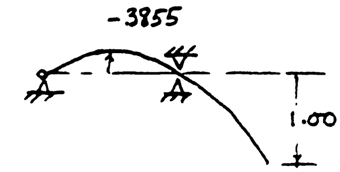
8-13 Cont

$$\bar{\lambda} = \frac{12.8\,EI}{\omega^2 \ell^3}$$

```
»[v,d]=eig(d)

v =

   -0.4605    0.3597
   -0.8876   -0.9331


d =

   56.2356         0
         0  237.0977
```

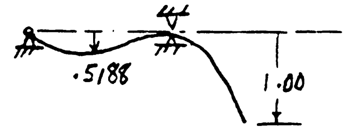
$\omega_1 = .2323\sqrt{\frac{EI}{\ell^3}}$

$\phi_1 = \begin{Bmatrix} -.3855 \\ 1.000 \end{Bmatrix}$



$\omega_2 = .4771\sqrt{\frac{EI}{\ell^3}}$

$\phi_2 = \begin{Bmatrix} .5188 \\ 1.000 \end{Bmatrix}$



8-14 Equation of motion is

$$m\begin{bmatrix} 3 & & \\ & 2 & \\ & & 1 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + k\begin{bmatrix} 6 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

```
»M=[ 3 0 0 ; 0 2 0 ; 0 0 1]

M =

     3     0     0
     0     2     0
     0     0     1

»K=[ 6 -2 0; -2 3 -1;0 -1 1]

K =

     6    -2     0
    -2     3    -1
     0    -1     1
```

8-14 cont

```
»d=inv(K)*M

d =

    0.7500    0.5000    0.2500
    0.7500    1.5000    0.7500
    0.7500    1.5000    1.7500

»[v,d]=eig(d)
```

$\bar{\lambda} = \left(\frac{K}{\omega^2 m}\right)$

```
v =

    0.2160   -0.6256    0.4337
    0.5376    0.6743    0.3636
    0.8150   -0.3924   -0.8245


d =

    2.9382         0         0
         0    0.3678         0
         0         0    0.6939
```

$\omega_1 = .5834\sqrt{K/m}$

$\omega_2 = 1.649\sqrt{K/m}$

$\omega_3 = 1.201\sqrt{K/m}$

8-15 The flexibility matrix is given in Example 6.1-3

$$[a] = \frac{\ell^3}{3EI}\begin{bmatrix} 27 & 14 & 4 \\ 14 & 8 & 2.5 \\ 4 & 2.5 & 1 \end{bmatrix} \qquad \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \leftarrow \text{free end}$$

Inverse of above matrix is

$$\frac{\ell^3}{3EI}\begin{bmatrix} 27 & 14 & 4 \\ 14 & 8 & 2.5 \\ 4 & 2.5 & 1 \end{bmatrix}^{-1} = \frac{3EI}{\ell^3}\begin{bmatrix} .5385 & -1.231 & .9231 \\ & 3.385 & -3.539 \\ & & 6.154 \end{bmatrix}$$

8-15 cont

```
»M=[ 1 0 0 ; 0 1 0 ; 0 0 1]

M =

     1     0     0
     0     1     0
     0     0     1

»K=[ .5385 -1.231 .9231 ; -1.231 3.385 -3.539; .9231 -3.539 6.154]

K =

    0.5385   -1.2310    0.9231
   -1.2310    3.3850   -3.5390
    0.9231   -3.5390    6.1540

»d=inv(K)*M
```

$\lambda = \frac{3EI}{\omega^2 \ell^3}$

```
d =

   27.1245   14.0695    4.0223
   14.0695    8.0387    2.5124
    4.0223    2.5124    1.0040

»[v,d]=eig(d)

v =
```

$$\begin{array}{ccc} \phi_1 & \phi_2 & \phi_3 \\ 0.4526 & 0.1737 & 0.8746 \\ -0.6822 & -0.5642 & 0.4651 \\ -0.5743 & 0.8071 & 0.1369 \end{array}$$

$\phi_1 = \begin{pmatrix} 1.00 \\ -1.504 \\ -1.269 \end{pmatrix}$ ← free end

$\phi_2 = \begin{pmatrix} 1.00 \\ -3.248 \\ 4.647 \end{pmatrix}$ $\phi_3 = \begin{pmatrix} 1.000 \\ .5316 \\ .1564 \end{pmatrix}$

```
d =

    0.8180         0         0
         0    0.1133         0
         0         0   35.2359
```

» $\omega_1 = 1.916\sqrt{\frac{EI}{\ell^3}}$ $\omega_3 = .2925\sqrt{\frac{EI}{\ell^3}}$

$\omega_2 = 5.146\sqrt{\frac{EI}{\ell^3}}$

8-20

$$3m\ddot{x}_1 = -4kx_1 + k(x_2 - x_1)$$
$$m\ddot{x}_2 = -k(x_2 - x_1) - kx_2$$

$$M = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \qquad K = \begin{bmatrix} 5 & -1 \\ -1 & 2 \end{bmatrix} \qquad Q = \begin{bmatrix} 2.2361 & -.44721 \\ 0 & 1.3416 \end{bmatrix}$$ see App C5

$$Q^{-1} = \begin{bmatrix} .44721 & .14907 \\ 0 & .74535 \end{bmatrix} \qquad [Q^{-T} M Q^{-1} - \bar{\lambda} I]Y = 0 \qquad \bar{\lambda} = \frac{k}{\omega^2 m}$$

$$Q^{-T} M Q^{-1} = \begin{bmatrix} .4472 & 0 \\ .1491 & .7454 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} .4472 & .1491 \\ 0 & .7454 \end{bmatrix} = \begin{bmatrix} .600 & .200 \\ .200 & .6222 \end{bmatrix} = \tilde{A}$$

standard form $[\tilde{A} - \bar{\lambda} I]Y = 0 \qquad Y = QX \quad$ or $\quad X = Q^{-1}Y$

8-21

$$K = k\begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \qquad Q = U = \begin{bmatrix} 1.732 & -.5774 & -.5774 \\ 0 & 1.2910 & -1.0328 \\ 0 & 0 & .7746 \end{bmatrix}$$

See App. C5

$$Q^{-1} = U^{-1} = \begin{bmatrix} .5773 & .2582 & .7746 \\ 0 & .7746 & 1.0328 \\ 0 & 0 & 1.2910 \end{bmatrix}$$

```
»k=[ 3 -1 -1 ; -1 2 -1 ; -1 -1 2]

k =

     3    -1    -1
    -1     2    -1
    -1    -1     2

»q=chol(k)

q =

    1.7321   -0.5774   -0.5774
         0    1.2910   -1.0328
         0         0    0.7746
```

8 – 16

$$M = J\begin{bmatrix}1 & & \\ & 1 & \\ & & 1\end{bmatrix} \qquad K = k\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix}$$

```
»k=[1 -1 0 ; -1 2 -1; 0 -1 2 ]

k =

     1    -1     0
    -1     2    -1
     0    -1     2

»d=inv(k)

d =

     3     2     1
     2     2     1
     1     1     1

»[v,d]=eig(d)

v =
```

ϕ_1	ϕ_2	ϕ_3
0.3280	-0.5910	0.7370
-0.7370	0.3280	0.5910
0.5910	0.7370	0.3280

```
d =
```

ω_1	ω_2	ω_3
0.3080	0	0
0	0.6431	0
0	0	5.0489

$$\bar{\lambda} = \frac{k}{\omega^2 J}$$

$$\omega_i = \frac{1}{\sqrt{\bar{\lambda}}}\sqrt{\frac{k}{J}}$$

$$\phi_1 = \begin{pmatrix}1.000\\ -2.247\\ 1.802\end{pmatrix} \leftarrow \text{free end}$$

$$\phi_2 = \begin{pmatrix}1.000\\ -.5550\\ -1.247\end{pmatrix} \qquad \phi_3 = \begin{pmatrix}1.000\\ 0.8019\\ .4450\end{pmatrix}$$

$$\omega_1 = 1.802\sqrt{\frac{k}{J}} \qquad \omega_2 = 1.247\sqrt{\frac{k}{J}}$$

$$\omega_3 = .445\sqrt{\frac{k}{J}}$$

8 – 17 Equation of motion

$$-\omega^2 m\begin{bmatrix}1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1\end{bmatrix}\begin{Bmatrix}y_1\\ y_2\\ y_3\\ y_4\end{Bmatrix} + \frac{T}{\ell}\begin{bmatrix}2 & -1 & 0 & 0\\ -1 & 2 & -1 & 0\\ 0 & -1 & 2 & -1\\ 0 & 0 & -1 & 2\end{bmatrix}\begin{Bmatrix}y_1\\ y_2\\ y_3\\ y_4\end{Bmatrix} = \begin{Bmatrix}0\\ 0\\ 0\\ 0\end{Bmatrix}$$

8-17 cont

$$\bar{\lambda} = \frac{T}{\omega^2 m l}$$

```
»K=[ 2 -1 0 0 ; -1 2 -1 0 ; 0 -1 2 -1; 0 0 -1 2]

K =

     2    -1     0     0
    -1     2    -1     0
     0    -1     2    -1
     0     0    -1     2

»d=inv(K)

d =

    0.8000    0.6000    0.4000    0.2000
    0.6000    1.2000    0.8000    0.4000
    0.4000    0.8000    1.2000    0.6000
    0.2000    0.4000    0.6000    0.8000

»[v,d]=eig(d)

v =

    0.3717    0.6015    0.6015   -0.3717
    0.6015    0.3717   -0.3717    0.6015
    0.6015   -0.3717   -0.3717   -0.6015
    0.3717   -0.6015    0.6015    0.3717


d =

    2.6180         0         0         0
         0    0.7236         0         0
         0         0    0.3820         0
         0         0         0    0.2764
```

$\omega_1 = .618$

$\omega_2 = 1.175 \sqrt{\frac{T}{ml}}$

$\omega_3 = 1.617 \sqrt{\frac{T}{ml}}$

$\omega_4 = .1902$

8-18

```
»x=[2 -1 ; -1 4]

x =

     2    -1
    -1     4

»U=chol(x)

U =

    1.4142   -0.7071
         0    1.8708

»U'*U
```

Check calculation

```
ans =

    2.0000   -1.0000
   -1.0000    4.0000
```

8-19

```
»x=[3 -1 ; -1 1]

x =

     3    -1
    -1     1

»U=chol(x)

U =

    1.7321   -0.5774
         0    0.8165
```

8-21 Cont

```
»inv(q)

ans =

    0.5774    0.2582    0.7746
         0    0.7746    1.0328
         0         0    1.2910
```

8-22

$$m\begin{bmatrix}3 & 0\\0 & 2\end{bmatrix}\begin{Bmatrix}\ddot{x}_1\\\ddot{x}_2\end{Bmatrix} + k\begin{bmatrix}3 & -1\\-1 & 2\end{bmatrix}\begin{Bmatrix}x_1\\x_2\end{Bmatrix} = \begin{Bmatrix}0\\0\end{Bmatrix}$$ Decompose M

Let $\begin{bmatrix}3 & 0\\0 & 2\end{bmatrix} = Q^TQ \qquad Q = \begin{bmatrix}\sqrt{3} & 0\\0 & \sqrt{2}\end{bmatrix} = \begin{bmatrix}1.7321 & 0\\0 & 1.414\end{bmatrix}$

$Q^{-1} = \begin{bmatrix}.5773 & 0\\0 & .7072\end{bmatrix} = Q^{-T} \qquad \left[-\left(\frac{\omega^2 m}{k}\right)I + Q^{-T}KQ^{-1}\right]Y = 0$

$$Q^{-T}KQ^{-1} = \begin{bmatrix}.5773 & 0\\0 & .707\end{bmatrix}\begin{bmatrix}3 & -1\\-1 & 2\end{bmatrix}\begin{bmatrix}.5773 & 0\\0 & .707\end{bmatrix} = \begin{bmatrix}1.00 & -.4082\\-.4082 & 1.00\end{bmatrix} = \tilde{A}$$

```
»m=[3 0 ; 0 2]

m =

     3     0
     0     2

»k=[3 -1 ; -1 2]

k =

     3    -1
    -1     2

»q=sqrt(m)

q =

    1.7321         0
         0    1.4142
```

8-22 cont

```
»a=inv(q)*k*inv(q)

a =

    1.0000   -0.4082
   -0.4082    1.0000

»[v,d]=eig(a)
```

$\lambda = \frac{\omega^2 m}{K}$

```
v =
```

$$\begin{array}{cc} \phi_2 & \phi_1 \\ -0.7071 & -0.7071 \\ 0.7071 & -0.7071 \end{array}$$

$\phi_2 = \begin{pmatrix} -.817 \\ 1.00 \end{pmatrix}$ $\phi_1 = \begin{pmatrix} .817 \\ 1.00 \end{pmatrix}$

```
d =

    1.4082         0
         0    0.5918
```

$\omega_2 = 1.187\sqrt{K/m}$

$\omega_1 = .7693\sqrt{K/m}$

8-23

$$\left[-\frac{\omega^2 m}{K}\begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{bmatrix} 4 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \right] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

Decompose K

see example 8.8-2

```
»m=[4 0 0 ; 0 2 0 ; 0 0 1]

m =

     4     0     0
     0     2     0
     0     0     1

»k=[4 -1 0 ; -1 2 -1 ; 0 -1 1]

k =

     4    -1     0
    -1     2    -1
     0    -1     1
```

8-25 cont

```
»a=chol(k)

a =

    2.0000   -0.5000         0
         0    1.3229   -0.7559
         0         0    0.6547

»ain=inv(a)

ain =

    0.5000    0.1890    0.2182
         0    0.7559    0.8729
         0         0    1.5275

»atild=ain'*m*ain

atild =

    1.0000    0.3780    0.4364
    0.3780    1.2857    1.4846
    0.4364    1.4846    4.0476

»[v,d]=eig(atild)

v =

    0.4232   -0.8944    0.1447
   -0.8516   -0.3381    0.4006
    0.3094    0.2928    0.9047

d =

    0.5585         0         0
         0    1.0000         0
         0         0    4.7749
```

$\bar{\lambda} = \frac{k}{\omega^2 m}$ $\qquad \omega = \frac{1}{\sqrt{\bar{\lambda}}}\sqrt{\frac{k}{m}}$

$\omega_1 = 1.3381\sqrt{k/m}$

$\omega_2 = 1.00\sqrt{k/m}$

$\omega_3 = .4576\sqrt{k/m}$

8-23 cont.

```
»x=qin*v

x =

   0.1182   -0.4472    0.3455
  -0.3737   -0.0000    1.0926
   0.4726    0.4472    1.3820
```

$$x = Q^{-1} y$$

where $\left[Q^{-T} m Q^{-1} - \bar{\lambda} I\right] y = 0$

8-24

```
»m=[4 1 0; 1 4 1; 0 1 2]

m =

     4     1     0
     1     4     1
     0     1     2

»k=[2 -1 0 ; -1 2 -1 ; 0 -1 1]

k =

     2    -1     0
    -1     2    -1
     0    -1     1

»q=chol(m)

q =

   2.0000    0.5000         0
        0    1.9365    0.5164
        0         0    1.3166
```

The matrix problem is

$$\left[-\lambda I + Q^{-T} k Q^{-1}\right] y = 0$$

$$\lambda = \frac{\omega^2 m}{k}$$

$$x = Q^{-1} y$$

8-24 cont

```
»qin=inv(q)

qin =

    0.5000   -0.1291    0.0506
         0    0.5164   -0.2025
         0         0    0.7596

»a=qin'*k*qin

a =

    0.5000   -0.3873    0.1519
   -0.3873    0.7000   -0.6668
    0.1519   -0.6668    0.9923
```

dynamic matrix

```
»[v,d]=eig(a)

v =

    0.4887    0.8165    0.3074
    0.7480   -0.2108   -0.6293
    0.4490   -0.5375    0.7138


d =

    0.0467         0         0
         0    0.5000         0
         0         0    1.6456

»x=qin*v

x =

    0.1705    0.4082    0.2711
    0.2953   -0.0000   -0.4695
    0.3410   -0.4082    0.5422

»
```

$x = Q^{-1} y$

8-25

$$[Q^{-T} m Q^{-1} - \bar{\lambda} I] y = 0 \qquad \bar{\lambda} = \frac{k}{\omega^2 m} \qquad x = Q^{-1} y$$

```
»k=[2 -1 0 ; -1 2 -1 ; 0 -1 1 ]

k =

     2    -1     0
    -1     2    -1
     0    -1     1

»m=[4 1 0 ; 1 4 1 ; 0 1 2]

m =

     4     1     0
     1     4     1
     0     1     2

»q=chol(k)

q =

    1.4142   -0.7071         0
         0    1.2247   -0.8165
         0         0    0.5774

»qin=inv(q)

qin =

    0.7071    0.4082    0.5774
         0    0.8165    1.1547
         0         0    1.7321

»a=qin'*m*qin

a =

    2.0000    1.7321    2.4495
    1.7321    4.0000    7.0711
    2.4495    7.0711   18.0000
```

8-25 cont

```
»[v,d]=eig(a)

v =

   -0.8165   -0.1494    0.5577
   -0.4714   -0.3851   -0.7934
    0.3333   -0.9107    0.2440


d =

    2.0000         0         0
         0   21.3923         0
         0         0    0.6077
```

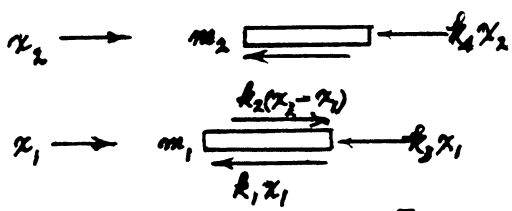
$\bar{\lambda}_i$

$\bar{\omega}_i = \frac{1}{\sqrt{\bar{\lambda}}} \sqrt{\frac{k}{m}}$

```
»x=qin*v

x =

   -0.5774   -0.7887    0.2113
    0.0000   -1.3660   -0.3660
    0.5774   -1.5774    0.4226
```

$X = Q^{-1} Y$

8 – 26



$$m_2 \ddot{x}_2 = -k_2(x_2 - x_1) - k_4 x_2$$

$$m_1 \ddot{x}_1 = -k_1 x_1 + k_2(x_2 - x_1) - k_3 x_1$$

$$\begin{bmatrix} m_1 & \\ & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2 + k_3) & -k_2 \\ -k_2 & (k_2 + k_4) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

8-26 Cont

```
»m=[ 1 0 ; 0 1]

m =

     1     0
     0     1

»k=[ 3 -1 ; -1 2 ]

k =

     3    -1
    -1     2

»[v,d]=eig(k)

v =

   -0.8507   -0.5257
    0.5257   -0.8507


d =

    3.6180         0
         0    1.3820
```

$\lambda = \frac{\omega^2 m}{k}$

8 - 21

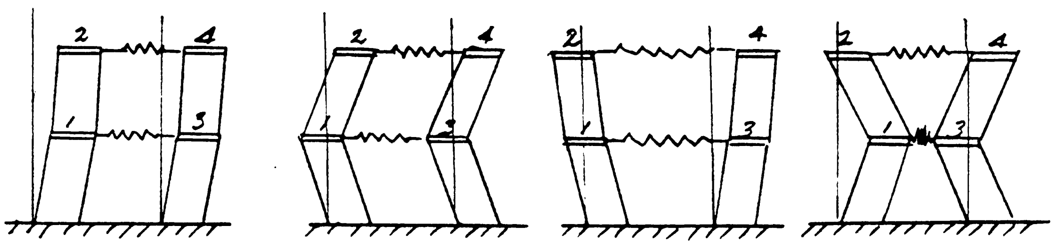
$$M = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} m. \qquad K = k\begin{bmatrix} 3 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

$$m\ddot{x}_1 = -k_1 x_1 + k(x_2 - x_1) + k(x_3 - x_1)$$

$$m\ddot{x}_2 = -k(x_2 - x_1) + k(x_4 - x_2)$$

$$m\ddot{x}_3 = -k x_3 + k(x_4 - x_3) - k(x_3 - x_1)$$

$$m\ddot{x}_4 = -k(x_4 - x_3) - k(x_4 - x_2)$$



ϕ_1 ϕ_3 ϕ_2 ϕ_4

```
»m=eye(4)

m =

     1     0     0     0
     0     1     0     0
     0     0     1     0
     0     0     0     1

»k=[3 -1 -1 0 ; -1 2 0 -1 ; -1 0 3 -1 ; 0 -1 -1 2]

k =

     3    -1    -1     0
    -1     2     0    -1
    -1     0     3    -1
     0    -1    -1     2
```

8-27 cont

```
»[v,d]=eig(k)
```

v =

ϕ_4	ϕ_2	ϕ_3	ϕ_1
0.6015	-0.3717	+0.6015	+0.3717
-0.3717	-0.6015	-0.3717	+0.6015
-0.6015	0.3717	-0.6015	+0.3717
+0.3717	0.6015	+0.3717	+0.6015

d =

2.6180	0	0	0
0	2.3820	0	0
0	0	4.6180	0
0	0	0	0.3820

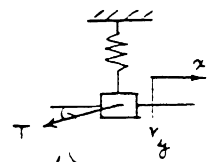
9-1

$$c = \sqrt{\frac{T}{\rho}} = \sqrt{\frac{444}{.372}} = 34.55 \text{ m/s}$$

9-2

$$f_n = \frac{n}{2\ell}\sqrt{\frac{T}{\rho}}, \qquad n = 1, 2, 3 \ldots$$

9-3



slope $\frac{dy}{dx}(\ell, t)$ is negative

gen. sol. $y(x,t) = \left(A \sin\frac{\omega x}{c} + B\cos\frac{\omega x}{c}\right)\sin\omega t$

At $x = 0$ $\quad y = 0 \quad \therefore B = 0$

At $x = \ell$ $\quad y = y(\ell, t)$ of spring mass

Vertical force $= -T\frac{dy}{dx}(\ell, t)$

$$= \left(-TA\frac{\omega}{c}\cos\frac{\omega\ell}{c}\right)\sin\omega t$$

$$= m\ddot{y} + ky$$

$$\therefore y(\ell) = \frac{-TA\frac{\omega}{c}\cos\frac{\omega\ell}{c}}{k - m\omega^2} = \frac{-TA\frac{\omega}{c}\cos\frac{\omega\ell}{c}}{m\omega_n^2\left(1 - \frac{\omega^2}{\omega_n^2}\right)}, \qquad \omega_n = \sqrt{\frac{k}{m}}$$

$$\text{or} \quad \tan\frac{\omega\ell}{c} = -\left(\frac{T}{k\ell}\right)\frac{\left(\frac{\omega\ell}{c}\right)}{\left[1 - \left(\frac{\omega\ell}{c}\right)^2\left(\frac{mc^2}{k\ell^2}\right)\right]}$$

9-4

$$y_1 = a\cos kx \sin\omega t, \qquad y_2 = a\cos\left(\frac{k}{x} + \frac{\pi}{2}\right)\sin\left(\omega t + \frac{\pi}{2}\right)$$

$$= -a\sin kx\cos\omega t$$

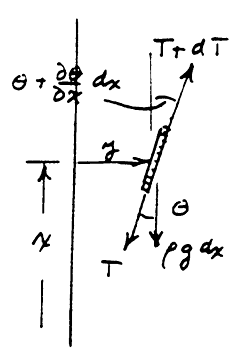
$$y = y_1 + y_2 = a\left[\cos kx\sin\omega t - \sin kx\cos\omega t\right]$$

$$= a\sin(\omega t - kx) = a\sin k\left(\frac{\omega}{k}t - x\right) \quad \therefore c = \frac{\omega}{k}$$

9-5

$$c = \sqrt{\frac{E}{\rho}} = \sqrt{\frac{200 \times 10^9}{7810}} = 5060 \text{ m/s} = 16{,}600 \text{ ft/s}$$

9-6 $\quad dT \cong \rho g\, dx$



$$(T + dT)\left(\theta + \frac{\partial\theta}{\partial x}dx\right) - T\theta = \rho\, dx\, \ddot{y}$$

$$\therefore T\frac{\partial\theta}{\partial x} + \rho g\,\theta = \rho\ddot{y} \qquad \theta = \frac{\partial y}{\partial x}, \quad T = \rho g x$$

$$\rho g x\frac{\partial^2 y}{\partial x^2} + \rho g\frac{\partial y}{\partial x} = \rho\ddot{y}$$

$$\ddot{y} = g\left(x\frac{\partial^2 y}{\partial x^2} + \frac{\partial y}{\partial x}\right)$$

9-7

$$y = Y(x)\cos\omega t$$
$$\ddot{y} = -\omega^2 Y(x)\cos\omega t$$

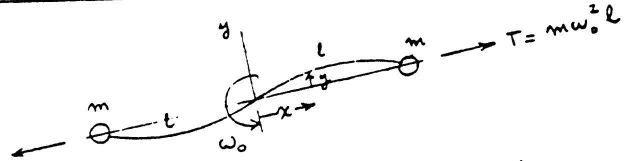
$$-\omega^2 Y(x) = g\left(x\frac{d^2Y(x)}{dx^2} + \frac{dY(x)}{dx}\right)$$

Let $z^2 = \frac{4\omega^2}{g}x \qquad dx = \frac{g}{4\omega^2}2z\,dz, \quad (dx)^2 = \left(\frac{g2z}{4\omega^2}\right)^2 dz^2$

$$-\omega^2 Y = g\left\{\frac{g}{4\omega^2}z^2\frac{d^2Y}{\frac{g^2z^2}{4\omega^2}dz^2} + \frac{2\omega^2}{gz}\frac{dY}{dz}\right\}$$

$$\therefore \quad \frac{d^2Y}{dz^2} + \frac{1}{z}\frac{dY}{dz} + Y = 0 \qquad \text{Bessel's D.E}$$

9-8



Assume mode shape as shown. Accel. at y is $\ddot{y} - y\omega_0^2$ in lateral direction

$$T\frac{\partial^2 y}{\partial x^2} = \rho(\ddot{y} - y\omega_0^2) \qquad \text{Let} \quad y = Y(x)e^{i\omega t}$$

$$\frac{d^2Y}{dx^2} + \left[\left(\frac{\omega}{c}\right)^2 + \left(\frac{\omega_0}{c}\right)^2\right]Y(x) = 0 \qquad c = \sqrt{\frac{T}{\rho}}$$

$Y(0) = 0 \quad \therefore \quad B = 0 \quad$ and $\quad Y(x) = A\sin\Omega x \neq 0$

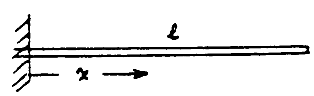
where $\Omega = \sqrt{\left(\frac{\omega}{c}\right)^2 + \left(\frac{\omega_0}{c}\right)^2}$

$Y(l) = 0 \quad \therefore \quad \sin\Omega l = 0$

$$\sqrt{\left(\frac{\omega}{c}\right)^2 + \left(\frac{\omega_0}{c}\right)^2}\, l = \pi \qquad \omega^2 = \left(\frac{\pi c}{l}\right)^2 - \omega_0^2$$

$$\therefore \quad \underline{\underline{\omega^2 = \left(\frac{\pi}{l}\right)^2\left(\frac{m\omega_0^2 l}{\rho}\right) - \omega_0^2}}$$

9-9



$$u = \left(A\sin\frac{\omega x}{c} + B\cos\frac{\omega x}{c}\right)\sin\omega t$$

$u(0) = 0 \quad \therefore B = 0$

$$\sigma = E\left(\frac{du}{dx}\right)_{x=l} = 0 \quad \therefore \quad \frac{\omega}{c}\cdot\cos\frac{\omega l}{c} = 0$$

$$\frac{\omega l}{c} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \;\cdots = \left(n + \tfrac{1}{2}\right)\pi \qquad n = 0, 1, 2 \cdots$$

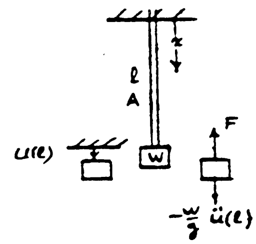
$$f_n = \frac{\omega_n}{2\pi} = \left(n + \tfrac{1}{2}\right)\frac{c}{2l}, \quad n = 0, 1, 2, \cdots$$

9-10

$$u = C_1 \sin\frac{\omega x}{c} \sin\omega t$$

$$F = A\sigma = AE\frac{\partial u}{\partial x} = AEC_1 \frac{\omega}{c}\cos\frac{\omega x}{c}\sin\omega t$$

$$= -\frac{W}{g}\ddot{u}(\ell)$$

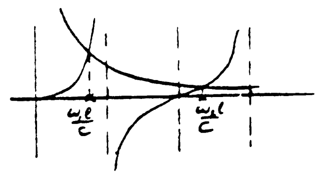


$$\therefore\ AE\frac{\omega}{c}\cos\frac{\omega\ell}{c} = \omega^2\frac{W}{g}\sin\frac{\omega\ell}{c}$$

$$\frac{\omega\ell}{c}\tan\frac{\omega\ell}{c} = \frac{A\rho\ell}{W} = \frac{W_{rod}}{W_{end}}$$

9-11

From Prob. 9-10 $\quad \tan\frac{\omega\ell}{c} = \dfrac{\left(\frac{m_{rod}}{M}\right)}{\frac{\omega\ell}{c}}$



Let freq. eq. be $\omega_1 = n_1\sqrt{\frac{Eg}{\rho_w}}$

where n_1 depends on end cond.

ρ_w = weight density

ρ = mass "

$$\omega_1 = n_1\sqrt{\frac{EA}{\ell}\frac{g\ell^2}{\rho_w A\ell}} = n_1\ell\sqrt{\frac{EA}{\ell}\frac{g}{\rho_w A\ell}} = n_1\ell\sqrt{\frac{k}{m_{rod}}}$$

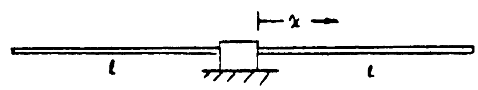
Let $r = \frac{m_{rod}}{M}$, then $\omega_1 = n_1\ell\sqrt{\frac{k}{rM}}$

Approx sol.

$$\omega_{appr.} \cong \sqrt{\frac{AE/\ell}{M + \frac{1}{3}m_{rod}}} = \sqrt{\frac{k}{M + \frac{r}{3}M}}$$

$$\frac{\omega_{appr.}}{\omega_1} = \sqrt{\frac{k}{M(1+\frac{1}{3}r)}}\sqrt{\frac{rM}{k}}\left(\frac{1}{n_1\ell}\right) = \frac{1}{\beta_1}\sqrt{\frac{3r}{3+r}}$$

9-12



$$u(0) = 0 \quad \therefore\ u(x) = A\sin\frac{\omega x}{c}$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=\ell} = 0 \quad \therefore\ \cos\frac{\omega\ell}{c} = 0 \qquad \frac{\omega\ell}{c} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}\cdots$$

$$\omega_1 = \frac{\pi}{2}\frac{c}{\ell} = \frac{\pi}{2\ell}\sqrt{\frac{Eg}{\rho_w}} = 2\pi(20{,}000)$$

$$\ell = \frac{1}{4\times 20{,}000}\sqrt{\frac{Eg}{\rho_w}} = \frac{10^3}{80{,}000}\sqrt{\frac{30\times 386}{0.31}} = 7.64''$$

Note $\rho = \frac{\rho_w}{g}$ = mass density

9-13

$$m\frac{\partial^2 u}{\partial t^2} = AE\frac{\partial^2 u}{\partial x^2} - \alpha\frac{\partial u}{\partial t} + \frac{p_0}{l}\,p(x)\,f(t)$$

$u = \sum_i \phi_i(x)\,q_i(t)$ subst into eq. above

$$m\sum_i \phi_i \ddot{q}_i = AE\sum_i \phi_i'' q_i - \alpha\sum_i \phi_i \dot{q}_i + \frac{p_0}{l}\,p(x)\,f(t)$$

mult. by $\phi_j\,dx$ and integrate over $x = 0$ to l

$$m\int_0^l \phi_j \sum_i \phi_i \ddot{q}_i\,dx = AE\int_0^l \phi_j \sum_i \phi_i'' q_i\,dx - \alpha\int_0^l \phi_j \sum_i \phi_i \dot{q}_i\,dx + \frac{p_0}{l}\int_0^l p(x)\phi_j\,dx\,f(t)$$

since ϕ_j and ϕ_i are orthogonal

$$\ddot{q}_j\int_0^l m\,\phi_j^2\,dx = AE\,q_j\int_0^l \phi_j\phi_j''\,dx - \alpha\,\dot{q}_j\int_0^l \phi_j^2\,dx + \frac{p_0}{l}f(t)\int_0^l p(x)\phi_j\,dx$$

$$\ddot{q}_j + 2\zeta\omega_j\dot{q}_j + \omega_j^2 q_j = \frac{p_0}{ml}f(t)\int_0^l p(x)\,\phi_j\,dx$$

with $b_j = \frac{1}{l}\int_0^l p(x)\,\phi_j\,dx$

$$\ddot{q}_j + 2\zeta\omega_j\dot{q}_j + \omega_j^2 q_j = \frac{p_0}{m}\,b_j\,f(t)$$

From Eq 4.2-1

$$q_j = \frac{p_0}{m}\,b_j\int_0^l f(t-\tau)\,e^{-\zeta\omega_j\tau}\sin\omega_j\sqrt{1-\zeta^2}\,\tau\,d\tau$$

$$u = \sum_j \phi_j q_j$$

9-14

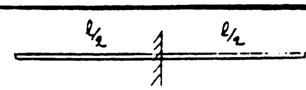
From Eq. 9.3-3 $\quad c = \sqrt{\frac{gG}{\rho_w}}$

for steel $\quad G = 12\times10^6$ psi

$\rho_w = 0.282$ lb/in³

$$c = 10^3\sqrt{\frac{12\times386}{0.282}} = 128{,}000 \text{ in/sec} = 10{,}660 \text{ ft/sec}$$

9-15



$\theta = A\sin\frac{\omega x}{c}$

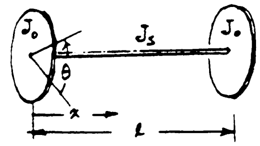
$\left(\frac{\partial\theta}{\partial x}\right)_{x=\frac{l}{2}} = 0 \quad \therefore \quad \cos\frac{\omega l}{2c} = 0$

$$\frac{\omega l}{2c} = \frac{\pi}{2},\ \frac{3\pi}{2},\ \frac{5\pi}{2}\ \cdots$$

$$\omega_n = \frac{\pi c}{l},\ \frac{3\pi c}{l},\ \cdots = (2n-1)\frac{c}{l} \qquad n = 1, 2, 3\cdots$$

9–16 $\quad J_0 \ddot{\theta}_{x=0} = G I_p \left(\frac{d\theta}{dx}\right)_{x=0}$

$$J_0 \ddot{\theta}_{x=\ell} = -G I_p \left(\frac{d\theta}{dx}\right)_{x=\ell}$$



$$\theta = \left(A \sin\frac{\omega x}{c} + B \cos\frac{\omega x}{c}\right)\sin\omega t$$

$$-\omega^2 J_0 B = G I_p \frac{\omega}{c} A$$

$$-\omega^2 J_0\left[A\sin\frac{\omega\ell}{c} + B\cos\frac{\omega\ell}{c}\right] = -G I_p \frac{\omega}{c}\left[A\cos\frac{\omega\ell}{c} - B\sin\frac{\omega\ell}{c}\right]$$

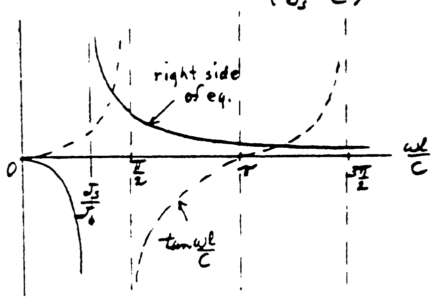
$$\therefore\ B = -\frac{G I_p}{\omega c J_0} A$$

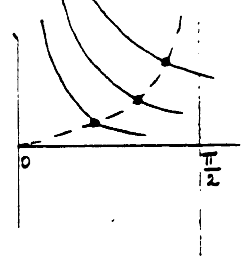
$$\sin\frac{\omega\ell}{c} - \frac{G I_p}{\omega c J_0}\cos\frac{\omega\ell}{c} = \frac{G I_p}{\omega c J_0}\left(\cos\frac{\omega\ell}{c} + \frac{G I_p}{\omega c J_0}\sin\frac{\omega\ell}{c}\right)$$

$$\frac{G I_p}{\omega c J_0} = \frac{G I_p}{\omega c J_0}\,\frac{\rho\ell}{g}\,\frac{g}{\rho\ell} = \frac{Gg}{\rho}\,\frac{J_s}{J_0}\,\frac{1}{\omega\ell c} = \frac{J_s}{J_0}\,\frac{c}{\omega\ell}$$

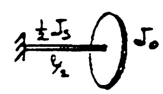
$$\therefore\left\{\tan\frac{\omega\ell}{c}\right\}\left\{1 + \left(\frac{J_s}{J_0}\,\frac{c}{\omega\ell}\right)^2\right\} = 2\,\frac{J_s}{J_0}\,\frac{c}{\omega\ell}$$

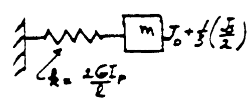
$$\tan\frac{\omega\ell}{c} = \frac{2\left(\frac{J_s}{J_0}\frac{\omega\ell}{c}\right)}{\left(\frac{J_0}{J_s}\frac{\omega\ell}{c}\right)^2 - 1} \qquad \frac{J_s}{J_0} < 1$$





When ends are free ($J_0 = 0$), $\frac{\omega\ell}{c} = \frac{\pi}{2}$. The effect of J_0 is to Lower nat. freq. If J_0/J_s is very large the right side of eq $\cong \dfrac{2}{\left(\frac{J_0}{J_s}\frac{\omega\ell}{c}\right)}$ Fundamental freq. has node at center





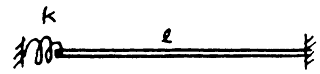
$$\omega_1 \cong \sqrt{\frac{2GI_p/\ell}{J_0 + \frac{1}{6}J_s}} = \sqrt{\frac{\frac{2Gg}{\rho}\,J_s/\ell^2}{J_0 + \frac{1}{6}J_s}} = \frac{c}{\ell}\sqrt{\frac{2J_s}{J_0 + \frac{1}{6}J_s}} = \frac{c}{\ell}\sqrt{\frac{2}{\frac{J_0}{J_s} + \frac{1}{6}}}$$

When $\frac{J_0}{J_s} = 0 \qquad \frac{\omega_1\ell}{c} = \sqrt{3}$

When $\frac{J_0}{J_s} = 5$, exact eq $\tan\frac{\omega\ell}{c} = \dfrac{10\,\frac{\omega\ell}{c}}{25\left(\frac{\omega\ell}{c}\right)^2 - 1}$ has root $\frac{\omega_1\ell}{c} = 0.622$

Approx eq. $\frac{\omega_1\ell}{c} = \sqrt{\frac{2}{5 + \frac{1}{6}}}$ gives 0.62

9-17



$$\theta = \left(A \sin \frac{\omega x}{c} + B \cos \frac{\omega x}{c}\right) \sin \omega t$$

$$\Theta(\ell) = 0 \quad \therefore \quad A \sin \frac{\omega \ell}{c} + B \cos \frac{\omega \ell}{c} = 0 , \qquad B = -A \tan \frac{\omega \ell}{c}$$

$$\theta = A\left(\sin \frac{\omega x}{c} - \tan \frac{\omega \ell}{c} \cos \frac{\omega x}{c}\right) \sin \omega t$$

Torque at $x = 0 \;=\; K\,\theta(0) = KA\left(-\tan \frac{\omega \ell}{c}\right)$

$$\left(\frac{d\theta}{dx}\right)_{x=0} = A \frac{\omega}{c}(1)$$

$$GI_p \left(\frac{d\theta}{dx}\right)_{x=0} = A \frac{\omega}{c} GI_p = -KA \tan \frac{\omega \ell}{c}$$

$$\therefore \tan \frac{\omega \ell}{c} = -\frac{\omega I_p G}{Kc} = -\frac{I_p G}{K \ell}\left(\frac{\omega \ell}{c}\right)$$

9-18

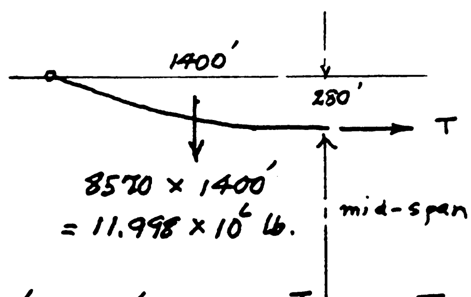
(a) Cables are non-horizontal

(b) Cable tension varies along length

(c) Stiffness of floor neglected

(d) Torsional mass moment of inertia greatly simplified

9-19



$$11.998 \times 10^6 \times 700' = 280T , \qquad T = 29.995 \times 10^6 \text{ lb.}$$

$$K_T = Tb^2 = 29.995 \times 10^6 \times 60^2 = 107,980 \times 10^6 \text{ lb ft}^2/\text{rad.}$$

$$\rho = \text{density} = \frac{8570}{32.2} = 266.1$$

Transverse vibration (see Ex. 9.4-1)

$$f_n = \frac{n}{2\ell}\sqrt{\frac{T}{\rho}} = \frac{n}{2 \times 2800}\sqrt{\frac{29.99 \times 10^6}{266.1}} = \frac{335.7}{5600} \times n$$

$$= 0.0600\, n = 3.59 \text{ c.p.m.} = f_1$$

$$f_2 = 7.19 \text{ c.p.m.}$$

9-19 Cont. Torsional vibration with node at mid-span ($n = 2$)

Estimate $J = \dfrac{8570}{32.2} \times k^2$

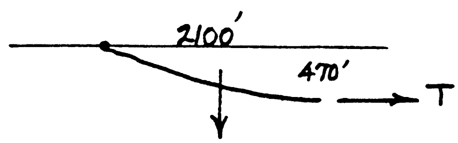
k = rad. of gyr. (assumed) $.73 \times 30' = 22'$

$\therefore J = 128,816$

$f_2 = \dfrac{1}{\ell}\sqrt{\dfrac{Tb^2}{J}}$ $\qquad \tau_2 = \ell\sqrt{\dfrac{J}{Tb^2}} = 2800\sqrt{\dfrac{128816}{107,982}} \times 10^{-3}$

$= 3.06$ sec. $\qquad f_2 = 0.327$

9-20



$$T = \frac{\overset{28,720 \times 2100}{60.312 \times 10^6} \times \frac{1}{2}(2100)}{470'} = 134.7 \times 10^6$$

$Tb^2 = 134.7 \times 90^2 \times 10^6 = 1091.4 \times 10^6$

= about 10 times torsional stiffness of New Tacoma bridge.

9-21

Rewrite equation as

$$J_0\ddot{\theta} + [c\dot{\theta} - f_2(v, \dot{\theta})] + [k\theta - f_1(v, \theta)] = 0$$

Depending on f_1 and f_2, the middle term may become negative $-c_{eq}\dot{\theta}$ creating a possibility of negative damping. The problem requires experimental research to determine f_1 and f_2, which can be very difficult.

9 – 22 $\quad y = A\cosh\beta x + B\sinh\beta x + C\cos\beta x + D\sin\beta x$

At $x=0$ and $x=l$, $\dfrac{d^2y}{dx^2} = \dfrac{d^3y}{dx^3} = 0$ resulting in four boundary equations

$$\text{at } x=0 \begin{cases} A + 0 - C + 0 = 0 & \therefore\ C = A \\ 0 + B + 0 - D = 0 & D = B \end{cases}$$

$$\text{at } x=l \begin{cases} A\cosh\beta l + B\sinh\beta l - C\cos\beta l - D\sin\beta l = 0 \\ A\sinh\beta l + B\cosh\beta l + C\sin\beta l - D\cos\beta l = 0 \end{cases}$$

$$-\frac{A}{B} = \frac{\sinh\beta l - \sin\beta l}{\cosh\beta l - \cos\beta l} = \frac{\cosh\beta l - \cos\beta l}{\sinh\beta l + \sin\beta l}$$

$\therefore$ freq. eq. becomes $\quad \underline{\underline{\cosh\beta l\cos\beta l = 1.0}}$

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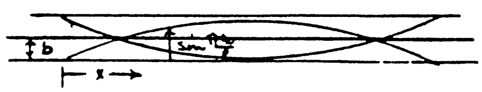
9 — 23



Momentum =

$$m\int_0^l \left(\sin\frac{\pi x}{l} - b\right)dx = 0 \qquad \text{gives} \quad b = \frac{2}{\pi}$$

$$T = \frac{1}{2}m\omega^2\int_0^l \left(\sin\frac{\pi x}{l} - b\right)^2 dx = \frac{1}{2}m\omega^2\left[\frac{l}{2} - \frac{8}{\pi^2}l + \frac{4}{\pi^2}l\right] = \frac{1}{2}m\omega^2 l\left(\frac{1}{2} - \frac{4}{\pi^2}\right)$$

$$U = \frac{1}{2}EI\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx = \frac{1}{2}EI\left(\frac{\pi}{l}\right)^4\int_0^l \frac{1}{2}\left(1-\cos\frac{2\pi x}{l}\right)dx = \frac{1}{2}EI\left(\frac{\pi}{l}\right)^4\frac{l}{2}$$

Equate T & U $\qquad \omega_1^2 = \dfrac{\pi^4}{\pi^2-8}\left(\dfrac{EI}{ml^4}\right) = 512\left(\dfrac{EI}{ml^4}\right)$

$\omega_1 = 22.6\sqrt{\dfrac{EI}{ml^4}} \qquad$ node at $\left(\sin\dfrac{\pi x}{l} - \dfrac{2}{\pi}\right) = 0$ or $\dfrac{x}{l} = 0.22$

---

9 — 24

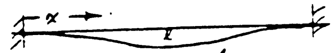
$$2\pi f_1 = 22.4\sqrt{\frac{EI}{ml^4}} = 2\pi\ 1690$$

$$m = 2\times 2\times 1\times\frac{153}{1732}\times\frac{1}{386} = 916\times 10^{-6}$$

$$I = \frac{2\times 2^3}{12} = \frac{4}{3} \qquad \frac{EI}{ml^4} = \left(\frac{2\pi\times 1690}{22.4}\right)^2 = 224{,}000$$

$$E = \frac{224{,}000\times 916\times 10^{-6}\times 12^4}{4/3} = 3{,}480{,}000 \ \ \text{lb/in}^2$$

9-25



Start with Eq. 9.5-12, At $x=0$ and $x=l$, $y=\frac{dy}{dx}=0$

$$\left.\begin{array}{l} A+0+C+0=0 \\ 0+B+0+D=0 \end{array}\right\} \quad \begin{array}{l} C=-A \\ D=-B \end{array}$$

$$A(\cosh\beta l-\cos\beta l)+B(\sinh\beta l-\sin\beta l)=0$$

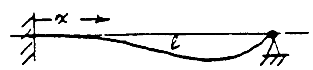
$$A(\sinh\beta l+\sin\beta l)+B(\cosh\beta l-\cos\beta l)=0$$

$$-\frac{A}{B}=\frac{\sinh\beta l-\sin\beta l}{\cosh\beta l-\cos\beta l}=\frac{\cosh\beta l-\cos\beta l}{\sinh\beta l+\sin\beta l}$$

or $\quad \cosh\beta l\cdot\cos\beta l=1$

9-26

Start with Eq. 9.5-12



At $x=0$, $y=\frac{dy}{dx}=0$ gives $C=-A$, $D=-B$

At $x=l$ $\quad y=\frac{d^2y}{dx^2}=0$

$$A(\cosh\beta l-\cos\beta l)+B(\sinh\beta l-\sin\beta l)=0$$

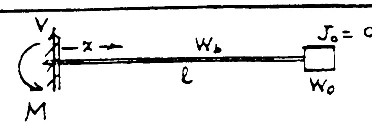
$$A(\cosh\beta l+\cos\beta l)+B(\sinh\beta l+\sin\beta l)=0$$

$$-\frac{A}{B}=\frac{\sinh\beta l-\sin\beta l}{\cosh\beta l-\cos\beta l}=\frac{\sinh\beta l+\sin\beta l}{\cosh\beta l+\cos\beta l}$$

$$\therefore\ \cosh\beta l\sin\beta l-\sinh\beta l\cos\beta l=0$$

or $\quad \tanh\beta l=\tan\beta l$

9-27



At $x=0$, $y=\frac{dy}{dx}=0$

$\therefore\ C=-A,\quad D=-B$

At $x=l$ $\quad -EI\frac{d^3y}{dx^3}=-V=\frac{W_0}{g}\ddot{y}(l)$

$$-\beta^3\left[A(\sinh\beta l-\sin\beta l)+B(\cosh\beta l+\cos\beta l)\right]$$
$$=-\frac{W_0}{g}\frac{\omega^2}{EI}\left[A(\cosh\beta l-\cos\beta l)+B(\sinh\beta l-\sin\beta l)\right)$$

9-27 Cont.

At $x = l$ $\qquad -M = -EI\frac{d^2y}{dx^2} = J_o\left(\frac{dy}{dx}\right) = 0$

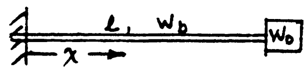
$$\beta^2\left[A(\cosh\beta l + \cos\beta l) + B(\sinh\beta l + \sin\beta l)\right] = 0$$

$$-\frac{A}{B} = \frac{(\cosh\beta l + \cos\beta l) - \frac{W_o\omega^2}{\beta^3 gEI}(\sinh\beta l - \sin\beta l)}{(\sinh\beta l - \sin\beta l) - \frac{W_o\omega^2}{\beta^3 gEI}(\cosh\beta l - \cos\beta l)} = \frac{(\sinh\beta l + \sin\beta l)}{(\cosh\beta l + \cos\beta l)}$$

$$\beta^2 = \omega\sqrt{\frac{wl}{gEIl}} = \omega\sqrt{\frac{W_b}{gEIl}}$$

$$\therefore\ (1 - \cosh\beta l\cos\beta l) = \frac{W_o}{W_b}\beta l(-\cosh\beta l\sin\beta l + \sinh\beta l\cos\beta l)$$

9-28



Assume static curve for concentrated load at end.

$y(x) = \frac{1}{EI}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right)$ Effective mass m_{eff} of uniform beam at end of beam is

$$T = \frac{1}{2}\int_0^l \dot{y}^2\,dm = \frac{1}{2}m_{eff}\left(\frac{l^3}{3EI}\right)^2, \qquad T = \frac{1}{2(EI)^2}\int_0^l\left(\frac{lx^2}{2} - \frac{x^3}{6}\right)^2 dx$$

$$= \frac{1}{2}\frac{ml}{(EI)^2}\cdot .0262\ l^6$$

$\therefore\ m_{eff} = 9 \times .0262\ ml = .235\ ml$

Total eff. mass $= \frac{1}{g}(.235\,W_b + W_o)$, $\quad \omega_1 = \sqrt{\frac{3EIg}{l^3(.235\,W_b + W_o)}}$

9-29

Additional mass to those of Prob 9-28 is

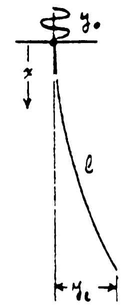
W_1 at $x = x_1$. The KE of W_1 is $\frac{1}{2}\frac{W_1}{g}\dot{y}(x_1)^2$

$\therefore$ Total effective mass is $\frac{1}{g}\left[.235\,W_b + W_o + W_1\left(\frac{y(x_1)}{y(l)}\right)^2\right]$

and

$$\omega_1 = \sqrt{\frac{3\,EI\,g}{\left[.235\,W_b + \left(\frac{y(x_1)}{y(l)}\right)^2 W_1 + W_o\right]}}$$

9-30



At $x=0$, $y=y_0$ $\therefore$ $y_0 = A + C$

At $x=0$ $\frac{d^2y}{dx^2} = 0$ $\therefore$ $C = A$

At $x=l$ $\frac{d^2y}{dx^2} = 0$

$$\beta^2\left[A(\cosh\beta l - \cos\beta l) + B\sinh\beta l - D\sin\beta l\right] = 0$$

At $x=l$ $\frac{d^3y}{dx^3} = 0$

$$\beta^3\left[A(\sinh\beta l + \sin\beta l) + B\cosh\beta l - D\cos\beta l\right] = 0$$

At $x=l$, $y = y_l$

$$y_l = A(\cosh\beta l + \cos\beta l) + B\sinh\beta l + D\sin\beta l$$

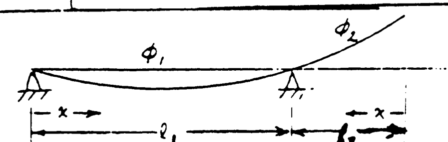
$$\frac{y_o}{y_l} = \frac{2A}{A(Ch+C) + BSh + Ds} = \frac{2A}{A(Ch+C) + BSh + A(Ch-C) + BSh}$$

where $Ch = \cosh\beta l$, $C = \cos\beta l$, $Sh = \sinh\beta l$, $S = \sin\beta l$

Also $y_0 = 2A$

$$\frac{y_o}{y_l} = \frac{\sinh\beta l\cos\beta l - \cosh\beta l\sin\beta l}{\sinh\beta l - \sin\beta l}$$

9-31



Boundary & intermediate cond.	$y_1(0) = 0$, $\frac{d^2y_1(0)}{dx^2} = 0$	$y_1(l) = 0$, $y_1''(l_1) = y_2''(l_2)$	$y_2''(0) = 0$, $y_2'''(0) = 0$

$y_1(0) = A_1 + C_1 = 0$ $\therefore C_1 = -A_1$

$y_1''(0) = A_1 - C_1 = 0$ $\therefore C_1 = +A_1$ $\Big\}$ $\therefore A_1 = C_1 = 0$

$$y_1 = B_1\sinh\beta x + D_1\sin\beta x$$

$$y_1(l_1) = B_1\sinh\beta l + D_1\sin\beta l = 0 \quad \therefore B_1 = -D_1\frac{\sin\beta l_1}{\sinh\beta l_1}$$

$$y_1 = D_1\left(\sin\beta x - \frac{\sin\beta l_1}{\sinh\beta l_1}\sinh\beta x\right) = \phi_1(x)$$

$y_2''(0) = A_2 - C_2 = 0$ $\therefore C_2 = A_2$

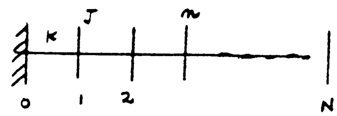
$y_2'''(0) = B_2 - D_2 = 0$ $\therefore D_2 = B_2$

$$y_2(l_2) = A_2(\cosh\beta l_2 + \cos\beta l_2) + B_2(\sinh\beta l_2 + \sin\beta l_2)$$

$$\therefore B_2 = -A\frac{\cosh\beta l_2 + \cos\beta l_2}{\sinh\beta l_2 + \sin\beta l_2}$$

$$y_2 = A_2\left\{(\cosh\beta x + \cos\beta x) - \left(\frac{\cosh\beta l_2 + \cos\beta l_2}{\sinh\beta l_2 + \sin\beta l_2}\right)(\sinh\beta x + \sin\beta x)\right\} = \phi_2(x)$$

9 – 32



$$-J\omega^2\theta_m = K(\theta_{m-1}-\theta_m) - K(\theta_m - \theta_{m-1})$$

$$\theta_{m-1} - 2\left(1-\frac{\omega^2 J}{2K}\right)\theta_m + \theta_{m+1} = 0 \qquad \text{Let } \theta_n = e^{\lambda n}$$

$$e^{-\lambda} - 2\left(1-\frac{\omega^2 J}{2K}\right) + e^{\lambda} = 0$$

$$\therefore \frac{\omega^2 J}{K} = 2(1-\cosh\lambda) \qquad \lambda = i\beta$$

$$= 2(1-\cos\beta) \qquad \theta_m = e^{i\beta m} = \cos\beta m + i\sin\beta m$$

Assume sol. $\theta_m = A\sin\beta m + B\cos\beta m$

Boundary $\theta_0 = 0 \quad \therefore B = 0$ and $\theta_m = A\sin\beta m$

Boundary N $\quad -\omega^2 J\theta_N = -K(\theta_N - \theta_{N-1})$

$$\left(1-\frac{J\omega^2}{K}\right)\theta_N = \theta_{N-1}$$

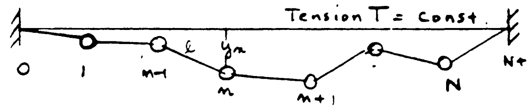
$$[1-2(1-\cos\beta)]\sin\beta N = \sin\beta(N-1)$$

Reduces to $\quad 2\sin\beta N\cos\beta = \sin\beta(N-1) + \sin\beta N$

$$2\cos\beta\left(N+\tfrac{1}{2}\right)\sin\frac{\beta}{2} = 0 \qquad \therefore \beta = 0, 2\pi, \dots$$

$$\omega_k = 2\sqrt{\frac{K}{J}}\,\sin\frac{(2k-1)\pi}{2(2N+1)} \qquad k = 1, 2, 3 \dots N$$

9-33



$$m\ddot{y}_m = -\frac{T}{\ell}(y_m - y_{m-1}) + \frac{T}{\ell}(y_{m+1} - y_m) \qquad \text{rearrange to}$$

$$y_{m+1} - 2\left(1 - \frac{\omega^2 m\ell}{2T}\right) y_m + y_{m-1} = 0$$

Let $y_m = e^{i\beta m}$ & subst. into above eq.

$$\cancel{e^{i\beta m}} e^{i\beta} - 2\left(1 - \frac{\omega^2 m\ell}{2T}\right)\cancel{e^{i\beta m}} + \cancel{e^{i\beta m}} e^{-i\beta} = 0$$

$$1 - \frac{\omega^2 m\ell}{2T} = \frac{e^{i\beta} + e^{-i\beta}}{2} = \cos\beta$$

$$\frac{\omega^2 m\ell}{2T} = 1 - \cos\beta = 2\sin^2\frac{\beta}{2}$$

freq. eq. $\omega = 2\sqrt{\frac{T}{m\ell}}\sin\frac{\beta}{2}$ Evaluate β at boundaries starting with gen. sol.

$$y_m = A\cos\beta m + B\sin\beta m$$

$y_0 = 0 \quad \therefore A = 0, \quad y_{N+1} = 0 \quad \therefore \sin\beta(N+1) = 0$

$$\beta(N+1) = 0, \pi, 2\pi, 3\pi \cdots = k\pi$$

$$\therefore \omega_k = 2\sqrt{\frac{T}{m\ell}}\sin\frac{k\pi}{2(N+1)} \qquad k = 1, 2, 3, \cdots$$

For $N = 2$

$$\omega_1 = 2\sqrt{\frac{T}{m\ell}}\sin\frac{\pi}{6} = \sqrt{\frac{T}{m\ell}}$$

$$\omega_2 = 2\sqrt{\frac{T}{m\ell}}\sin\frac{2\pi}{6} = \sqrt{\frac{3T}{m\ell}}$$

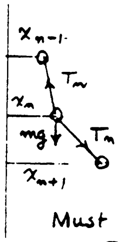
9-34

$$m\ddot{x}_m = k(x_{m+1} - x_m) - k_m(x_m - x_{m-1})$$

$$x_{m+1} - 2\left(1 - \frac{\omega^2 m}{k}\right)x_m + x_{m-1} = 0$$

This problem is identical to Prob 9-3- with **changes** in boundary conditions $\therefore \omega_k = 2\sqrt{\frac{k}{m}}\sin\frac{k\pi}{2(N+1)} \qquad k = 1, 2 \cdots$

9-35



$$m\ddot{x}_m = \frac{T_{m+1}}{\ell}(x_{m+1}-x_m) - \frac{T_m}{\ell}(x_m - x_{m-1})$$

$$T_m \cong T'_{m+1} + mg$$

$$x_{m+1} - 2\left(1 - \frac{\omega^2 m\ell}{T_m}\right)x_m + x_{m-1} = \frac{mg\ell}{T'_m}(x_{m+1} - x_m)$$

T_m changes ie $T_N = mg$, $T_{N-1} = 2mg$ etc.

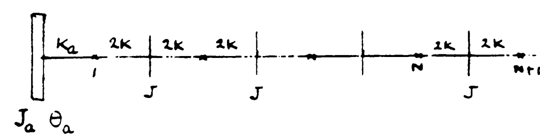
$\therefore$ Usual diff eq with const coef. cannot be used.

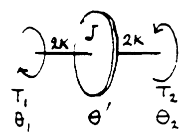
Must solve as a system of N eqs.

$$-\omega^2 m x_1 = -\frac{Nmg}{\ell}x_1 + \frac{(N-1)mg}{\ell}(x_2 - x_1)$$

$$-\omega^2 m x_N = -\frac{mg}{\ell}(x_N - x_{N-1})$$

9-36





$T_1 = 2K(\theta' - \theta_1)$ $\quad\longrightarrow\quad \begin{Bmatrix} T_1 \\ \theta_1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{-1}{2K} & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ \theta' \end{Bmatrix}$

$T_2 - T_1 = -\omega^2 J\theta'$ $\quad\longrightarrow\quad \begin{Bmatrix} T_1 \\ \theta' \end{Bmatrix} = \begin{bmatrix} 1 & \omega^2 J \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} T_2 \\ \theta' \end{Bmatrix}$

$T_2 = 2K(\theta_2 - \theta')$ $\quad\longrightarrow\quad \begin{Bmatrix} T_2 \\ \theta' \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{-1}{2K} & 1 \end{bmatrix}\begin{Bmatrix} T_2 \\ \theta_2 \end{Bmatrix}$

$$\therefore \begin{Bmatrix} T_1 \\ \theta_1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{-1}{2K} & 1 \end{bmatrix}\begin{bmatrix} 1 & \omega^2 J \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \frac{-1}{2K} & 1 \end{bmatrix}\begin{Bmatrix} T_2 \\ \theta_2 \end{Bmatrix}$$

$$= \begin{bmatrix} \left(1 - \frac{\omega^2 J}{2K}\right) & \omega^2 J \\ -\left(\frac{1}{K} - \frac{\omega^2 J}{4K^2}\right) & \left(1 - \frac{\omega^2 J}{2K}\right) \end{bmatrix}\begin{Bmatrix} T_2 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} A & B \\ C & A \end{bmatrix}\begin{Bmatrix} T_2 \\ \theta_2 \end{Bmatrix}$$

9–36 cont.

Note that $A^2 - BC = 1$ $\qquad B = \dfrac{A^2 - 1}{C}$

Let $A = \left(1 - \dfrac{\omega^2 J}{2k}\right) = \cosh\lambda$

$C = -\left(\dfrac{1}{k} - \dfrac{\omega^2 J}{4k}\right) = \dfrac{1}{z}\sinh\lambda$

$B = \omega^2 J = \dfrac{\cosh^2\lambda - 1}{\frac{1}{z}\sinh\lambda} = z\sinh\lambda$

$$\begin{Bmatrix} T_1 \\ \theta_1 \end{Bmatrix} = \begin{bmatrix} \cosh\lambda & z\sinh\lambda \\ \frac{1}{z}\sinh\lambda & \cosh\lambda \end{bmatrix} \begin{Bmatrix} T_2 \\ \theta_2 \end{Bmatrix}$$

If replaced by exponentials it can be proved that

$$\begin{bmatrix} \cosh\lambda & z\sinh\lambda \\ \frac{1}{z}\sinh\lambda & \cosh\lambda \end{bmatrix}^n = \begin{bmatrix} \cosh n\lambda & z\sinh n\lambda \\ \frac{1}{z}\sinh n\lambda & \cosh n\lambda \end{bmatrix}$$

$$\therefore \begin{Bmatrix} T_1 \\ \theta_1 \end{Bmatrix} = \begin{bmatrix} \cosh n\lambda & z\sinh n\lambda \\ \frac{1}{z}\sinh n\lambda & \cosh n\lambda \end{bmatrix} \begin{Bmatrix} T_{n+1} \\ \theta_{n+1} \end{Bmatrix}$$

From $A = \left(1 - \dfrac{\omega^2 J}{2k}\right) = \cosh\lambda$

$\omega^2 J = B = 2k(1 - \cosh\lambda) = -4k\sinh^2\dfrac{\lambda}{2}$

$\therefore$ freq. eq. $= \ \omega = 2\sqrt{\dfrac{k}{J}}\sinh\dfrac{\lambda}{2}$

where λ must be determined from boundary

Boundary conditions

$$\theta_1 - \theta_a = \frac{T_1}{k_a} = -\frac{\omega^2 J_a \theta_a}{k_a}, \qquad T_{N+1} = 0$$

$$\therefore T_1 = z\sinh N\lambda \cdot \theta_{N+1} = -\omega^2 J_a \theta_a$$

$$\theta_1 = \cosh N\lambda \cdot \theta_{N+1} = \left(1 - \frac{\omega^2 J_a}{k_a}\right)\theta_a$$

Divide $\qquad \dfrac{z\sinh N\lambda}{\cosh N\lambda} = \dfrac{-\omega^2 J_a}{\left(1 - \frac{\omega^2 J_a}{k_a}\right)}$

$$\frac{2k(1-\cosh\lambda)}{\sinh\lambda} \cdot \frac{\sinh N\lambda}{\cosh N\lambda} = \frac{-\omega^2 J_a}{\left(1 - \frac{\omega^2 J_a}{k_a}\right)}$$

$$-2k\left[\sinh N\lambda\cosh\lambda - \sinh N\lambda\right] = \frac{4k\frac{J_a}{J}\sinh^2\frac{\lambda}{2}}{1 + \frac{4k J_a}{k_a J}\sinh^2\frac{\lambda}{2}} \cdot \sinh\lambda\cosh N\lambda$$

$$(-2\sinh N\lambda \cdot \cosh\lambda + 2\sinh N\lambda)\left(1 - 4\frac{k}{k_a}\frac{J_a}{J}\sinh^2\frac{\lambda}{2}\right) = 2\frac{J_a}{J}\sinh^2\frac{\lambda}{2}\sinh\lambda\cosh N\lambda$$

$$(-\sin N\beta\cos\beta + \sin N\beta)\left(1 + 4\frac{k}{k_a}\frac{J_a}{J}\sin^2\frac{\beta}{2}\right) = -2\frac{J_a}{J}\sin^2\frac{\beta}{2}\sin\beta\cos N\beta$$

solve for β & subst. into freq. eq

$$\omega = 2\sqrt{\frac{k}{J}}\sin\frac{\beta}{2}$$

This problem can also be solved by using stations at the disks instead of at points midway between disks. The equations

$$\theta_n = A\sin\beta n + B\cos\beta n, \qquad n = 0, 1, 2 \cdots N$$

can be used provided $k_0 = \dfrac{1}{\frac{1}{k_a} + \frac{1}{2k}}$ is used.

9-37 With $n = 0$ fixed, solution is

$$X_m = B \sin \beta m$$

At top of the bldg. $m \ddot{x}_N = -k(x_N - x_{N-1}) - K_N x_N$

$$\text{or } X_{N-1} = \left(1 + \frac{K_N}{k} - \frac{m\omega^2}{k}\right) X_N$$

$$\therefore \sin \beta(N-1) = \left[1 + \frac{K_N}{k} - 2(1-\cos\beta)\right] \sin \beta N$$

$$\sin\beta(N-1) - \sin\beta N + 2(1-\cos\beta)\sin\beta N = \frac{K_N}{k} \sin\beta N$$

$$\sin\beta N\cos\beta - \cos\beta N\sin\beta - \sin\beta N + 2\sin\beta N - 2\sin\beta N\cos\beta = \frac{K_N}{k}\sin\beta N$$

$$-\sin\beta N\cos\beta - \cos\beta N\sin\beta + \sin\beta N = \frac{K_N}{k}\sin\beta N$$

$$\therefore -\sin\beta(N+1) + \sin\beta N = \frac{K_N}{k}\sin\beta N$$

$$2\cos\beta\left(N+\tfrac{1}{2}\right)\cdot\sin\frac{\beta}{2} = 2\left[\cos\beta N\cos\frac{\beta}{2}\sin\frac{\beta}{2} - \sin\beta N\sin^2\frac{\beta}{2}\right] = \left[\cos\beta N\sin\beta - \sin\beta N(1-\cos\beta)\right]$$

$$= \cos\beta N\sin\beta - \sin\beta N + \sin\beta N\cos\beta$$

$$\therefore \underline{\underline{-2\cos\beta\left(N+\tfrac{1}{2}\right)\cdot\sin\frac{\beta}{2} = \frac{K_N}{k}\sin\beta N}}$$

9-38 System is same as those of FIG. P9-33 P9-34

$$\therefore \omega_n = 2\sqrt{\frac{k}{m}} \sin\frac{n\pi}{2(N+1)}$$

9-39 Let h = height between stories

$$m\ddot{y}_n = -k\left[(y_n - nh\theta) - (y_{n-1} - (n-1)h\theta)\right] + k\left[(y_{n+1} - (n+1)h\theta) - (y_n - nh\theta)\right]$$

$$= -k\left[y_n - y_{n-1} - h\theta\right] + k\left[y_{n+1} - y_n - h\theta\right]$$

$$= k\left[y_{n+1} - 2y_n + y_{n-1}\right]$$

$\therefore$ for harmonic motion

$$Y_{n+1} - 2\left(1 - \frac{m\omega^2}{2k}\right)Y_n + Y_{n-1} = 0$$

gen. sol. $\quad Y_n = Y_0 \cos\beta n + B\sin\beta n$

boundary $\quad Y_N = Y_0\cos\beta N + B\sin\beta N$

$$B = \frac{Y_N - Y_0\cos\beta N}{\sin\beta N}$$

Gen sol. becomes

$$Y_n = Y_0\frac{\sin\beta(N-n)}{\sin\beta N} + Y_N\frac{\sin\beta n}{\sin\beta N}$$

boundary eq. for N^{th} mass

$$m\ddot{y}_N = -k\left[y_N - y_{N-1} + h\theta\right]$$

$$-\omega^2 m Y_N = -k\left[Y_N - Y_{N-1} + h\theta\right]$$

9-39 Cont.

$$\left(1 - \frac{\omega^2 m}{k}\right) Y_N = Y_{N-1} - h\theta$$

Torque Eq.

$$\sum_{n=1}^{N} n h (m \ddot{y}_N) - K_\theta \theta = (N+1) m \rho^2 \ddot{\theta}$$

$$-\omega^2 m h \sum_{n=1}^{N} n Y_n - K_\theta \theta = -(N+1) m\rho^2 \omega^2 \theta$$

or

$$\omega^2 m h \sum_{n=1}^{N} n Y_n - \left(K_\theta - (N+1) m\rho^2\omega^2\right)\theta = 0$$

9-40

$$\beta_k = \frac{(2k-1)\pi}{(2N+1)}$$

$$\omega_k = 2\sqrt{\frac{k}{m}} \sin\frac{(2k-1)\pi}{2(2N+1)}$$

$N =$ DOF

$k =$ mode number

Then $X_n = B \sin \beta_k n$. where $n =$ floor number

9-41

The stiffness of the cantilever beam is $\frac{3EI}{l^3}$, i.e., in the absence of the spring k, $F = \frac{3EI}{l^3} y$.

In the antisymmetric mode, where $y_1 = -y_2 = y$, the spring force is $k(y_1 - y_2) = 2ky$

$$\therefore \quad F = \frac{3EI}{l^3} y + 2ky \quad \text{(For each beam)}$$

$$= \left(\frac{3EI}{l^3} + 2k\right) y = k_{eq} y$$

In the absence of the spring force $\omega_1 = 3.52\sqrt{\frac{EI}{\rho l^4}}$

With the spring in the antisymmetric mode

$$\omega_1 = \sqrt{\frac{3EI/l^3 + 2k}{3EI/l^3}} * 3.52\sqrt{\frac{EI}{\rho l^4}}$$

$$= 3.52\sqrt{\frac{EI}{\rho l^4}\left(1 + \frac{2}{3}\frac{kl^3}{EI}\right)}$$

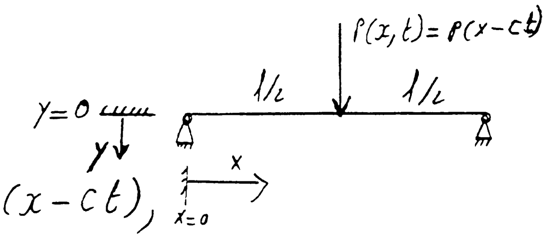
9-42

Euler equation

$(y = y(x,t))$

$$EI\frac{\partial^4 y}{\partial x^4} + m\frac{\partial^2 y}{\partial t^2} = p(x,t) = p(x-ct),$$

$$y(0,t) = y(l,t) = 0\ , \quad \left.\frac{\partial^2 y}{\partial x^2}\right|_{x=0} = \left.\frac{\partial^2 y}{\partial x^2}\right|_{x=l} = 0 \quad \text{for all } t$$



Let $y(x,t) = y(x-ct)$

$$\therefore\ EI\frac{\partial^4 y(x-ct)}{\partial x^4} + m\frac{\partial^2 y(x-ct)}{\partial t^2} = p(x-ct)$$

Let $z = x - ct$

$$\therefore\ EI\frac{\partial^4 y}{\partial z^4} + mc^2\frac{\partial^2 y}{\partial z^2} = p(z)$$

$$y(0) = y(l) = 0\ , \quad \left.\frac{\partial^2 y}{\partial z^2}\right|_{z=0} = \left.\frac{\partial^2 y}{\partial z^2}\right|_{z=l} = 0$$

Let $F = \frac{\partial^2 y}{\partial z^2}$

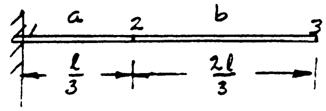
$$\therefore\ EIF'' + mc^2 F = p(z)$$

Solve for $F(z)$ then integrate twice to get $y(z) = y(x-ct)$.

9 - 43

This problem is similar to example 9.5-1. Here, the fact that the ball hits the bat, which acts as a cantilever beam, changes the boundary conditions at l. The boundary conditions do not change the natural frequencies, which are given by equation (9.5-13)

10-1 (see Ex. 10.1-1)



$$M_a = \frac{1}{3}M \qquad k_a = \frac{EA}{\ell/3} = \frac{6}{2}\frac{EA}{\ell}$$

$$M_b = \frac{2}{3}M \qquad k_b = \frac{EA}{2\ell/3} = \frac{3}{2}\frac{EA}{\ell}$$

with $u_1 = 0$

$$\text{Mass Matrix} = \frac{1}{6}\begin{bmatrix} 2(M_a+M_b) & M_b \\ M_b & 2M_b \end{bmatrix} \qquad \text{Ex. 10.1-1}$$

$$\text{Stiffness Matrix} = \begin{bmatrix} (k_a+k_b) & -k_b \\ -k_b & k_b \end{bmatrix}$$

Diff. Eq. of Motion with $\ddot{u} = -\omega^2 u$

$$\left\{ \frac{-\omega^2 M}{3\times 6}\begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} + \frac{EA}{2\ell}\begin{bmatrix} 9 & -3 \\ -3 & 3 \end{bmatrix} \right\}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{Let } \lambda = \frac{\omega^2 M}{18}\cdot\frac{2\ell}{EA} = \frac{\omega^2 M\ell}{9EA}$$

Characteristic Eq.

$$\begin{vmatrix} (9-6\lambda) & -(3+2\lambda) \\ -(3+2\lambda) & (3-4\lambda) \end{vmatrix} = 0 \quad \longrightarrow \quad \lambda^2 - 3.30\lambda + .90 = 0$$

$$\lambda = \begin{cases} .30 \\ 3.00 \end{cases}$$

$$\omega^2 = 9\times\begin{cases} .3 \\ 3.00 \end{cases} \qquad \begin{aligned} \omega_1 &= 1.643\sqrt{EA/M\ell} \\ \omega_2 &= 5.196\sqrt{EA/M\ell} \end{aligned} \qquad \text{Exact} \rightarrow \begin{cases} 1.5708 \\ 4.7124 \end{cases}$$

$\therefore$ 1st mode is 4.6% high

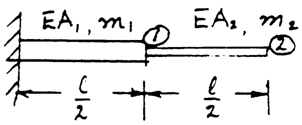
2nd mode is 10.3% high

With sta. at $\frac{\ell}{2}$, $\lambda^2 - \frac{10}{7}\lambda + \frac{1}{7} = 0$

$$\omega_1 = 1.6115\sqrt{\frac{EA}{M\ell}} = 1.025\times\text{Exact} = 2.6\%\text{ high}$$

$$\omega_2 = 5.6293\sqrt{\frac{EA}{M\ell}} = 1.1946\times\text{ "} = 19.5\%\text{ high}$$

10 − 2



$EA_1 = 2\,EA_2$

$m_1 = 2\,m_2$

$k_2 = \frac{2EA_2}{\ell}$

$$\left[-\frac{\omega^2}{6}\begin{bmatrix}2(M_a+M_b) & M_b\\ M_b & 2M_b\end{bmatrix} + \begin{bmatrix}(k_a+k_b) & -k_b\\ -k_b & k_b\end{bmatrix}\right]\begin{Bmatrix}u_1\\u_2\end{Bmatrix} = \begin{Bmatrix}0\\0\end{Bmatrix}$$

$$\left[-\frac{\omega^2 M_2}{6}\begin{bmatrix}2(2+1) & 1\\ 1 & 2\end{bmatrix} + \frac{2EA_2}{\ell}\begin{bmatrix}(2+1) & -1\\ -1 & 1\end{bmatrix}\right]\begin{Bmatrix}u_1\\u_2\end{Bmatrix} = \begin{Bmatrix}0\\0\end{Bmatrix}$$

Let $\lambda = \frac{\omega^2 M_2}{6}\left(\frac{\ell}{2EA}\right) = \left(\frac{\omega^2 M_2 \ell}{12EA_2}\right)$ $\therefore$ $\left|-\lambda\begin{bmatrix}6 & 1\\ 1 & 2\end{bmatrix} + \begin{bmatrix}3 & -1\\ -1 & 1\end{bmatrix}\right| = 0$

$$\begin{vmatrix}(3-6\lambda) & -(1+\lambda)\\ -(1+\lambda) & (1-2\lambda)\end{vmatrix} = 0 \qquad \lambda^2 - 1.2727\lambda + .1818 = 0$$

$$\lambda_{1,2} = \begin{cases}.1640\\ 1.1088\end{cases}$$

$\omega_1 = 1.4029\sqrt{\frac{EA_2}{M_2\ell}}$

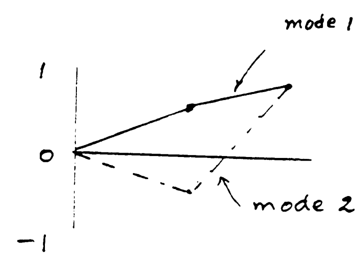
$\omega_2 = 3.6477\sqrt{\frac{EA_2}{M_2\ell}}$

Mode shapes $\quad (3-6\lambda)u_1 = (1+\lambda)u_2$

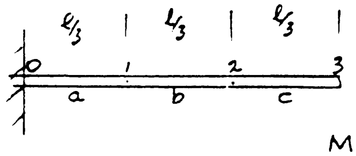
or $(1+\lambda)u_1 = (1-2\lambda)u_2$

$\left(\frac{u_1}{u_2}\right)_1 = \frac{(1-2\lambda)}{(1+\lambda)} = .5773$

$\left(\frac{u_1}{u_2}\right)_2 = -.5258$



10 − 3



$$\longleftrightarrow \quad u = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$M_a = M_b = M_c = \frac{M}{3}$$

$$k_a = k_b = k_c = \frac{EA_a}{\ell/3} = 3\frac{EA}{\ell}$$

Mass matrix

$$\left[\frac{M_a}{6}\begin{bmatrix} 2 & 1 & & \\ 1 & 2+2 & 1 & \\ & 1 & 2+2 & 1 \\ & & 1 & 2 \end{bmatrix}\right]$$

Stiffness matrix

$$k_a\left[\begin{matrix} 1 & -1 & & \\ -1 & 1+1 & -1 & \\ & -1 & 1+1 & -1 \\ & & -1 & 1 \end{matrix}\right]$$

$$\frac{M_a}{6}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_0 \\ \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} + k_a\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$u_0 = 0 \therefore$ cross out as above

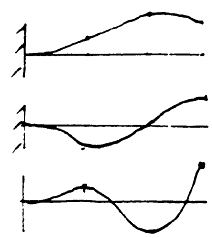
$$\frac{M}{18}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} + \frac{3EA}{\ell}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Computer solution: $\lambda = \dfrac{M\ell\omega^2}{54EA}$

		Exact	
$\omega_1 = 1.522\sqrt{\dfrac{EA}{M\ell}}$	$\left(\dfrac{EA}{M\ell} = \dfrac{1}{\ell}\sqrt{\dfrac{E}{\rho}}\right)$	$1.571\ \dfrac{1}{\ell}\sqrt{\dfrac{E}{\rho}}$	$\therefore$ 3% low
$\omega_2 = 5.1545$ "		4.712 "	9% high
$\omega_3 = 9.4329$ "		7.85 "	20% high

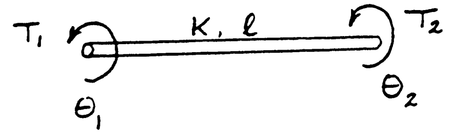
Mode Shapes

$$\begin{Bmatrix} 0 \\ .415 \\ .728 \\ .544 \end{Bmatrix}_1 \quad \begin{Bmatrix} 0 \\ -.811 \\ .0268 \\ .583 \end{Bmatrix}_2 \quad \begin{Bmatrix} 0 \\ .410 \\ -.684 \\ .602 \end{Bmatrix}_3$$



10 – 4 With linear twist, the problem is identical to that of the longitudinal vibration.

Torsional element of length ℓ



$$\text{Mass} = \rho A \ell \qquad \text{Stiffness} = \frac{I_p G}{\ell} \text{ of length } \ell$$

$$J_p = \frac{MR^2}{2} = \text{mass polar moment of inertia}$$

$$\text{Mass matrix} = \frac{\rho A \ell}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \text{Stiffness} = \frac{I_p G}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

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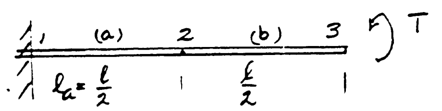
10 – 5



Torsion

$$M_a \frac{R^2}{2} = J_a$$

Let J = mass polar moment of inertia for entire rod

Then $J_a = \frac{1}{2} J$, $\quad K_a = \frac{I_p G}{\ell_a} = \frac{2 I_p G}{\ell}$

$$\frac{J_a}{6}\begin{bmatrix} \cancel{2} & \cancel{1} & \cancel{0} \\ \cancel{1} & 4 & 1 \\ \cancel{0} & 1 & 2 \end{bmatrix}\begin{Bmatrix} \cancel{\ddot{\theta}_1} \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} + K_a \begin{bmatrix} \cancel{1} & \cancel{-1} & \cancel{0} \\ \cancel{-1} & 2 & -1 \\ \cancel{0} & -1 & 1 \end{bmatrix}\begin{Bmatrix} \cancel{\theta_1} \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Units: $J = \text{mass in}^2$, $\quad I_p = \text{in}^4$, $\quad G = \text{lb/in}^2$, $\quad \ell = \text{in}$.

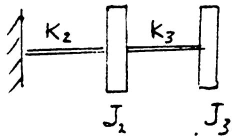
Let $\lambda = \dfrac{\omega^2 J_a}{6 K_a}$ $\qquad J_a = J/2$, $\qquad K_a = 2K$

Same prob. as longitudinal vibr.

$$\omega = \begin{cases} 1.6114 \sqrt{\dfrac{K}{J}} \\ 5.629 \quad '' \end{cases} \qquad \text{where } K = \frac{I_p G}{\ell} , \quad J = M\frac{R^2}{2}$$

$$\frac{K}{J} = \frac{2 I_p G}{M \ell R^2} = \frac{GA}{M\ell}$$

10-6 Lumped mass torsional system



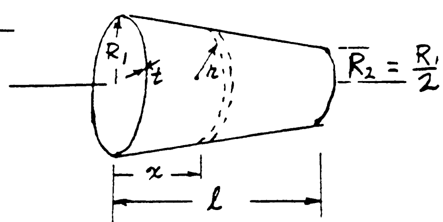
$$\begin{bmatrix} J_2 & 0 \\ 0 & J_3 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} + \begin{bmatrix} (K_2+K_3) & -K_3 \\ -K_3 & K_3 \end{bmatrix} \begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} [(K_2+K_3) - \omega^2 J_2] & -K_3 \\ -K_3 & [K_3 - \omega^2 J_3] \end{bmatrix} \begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \therefore \text{ same form}$$

The problem here is assigning proper values of J_2 and J_3, to be equivalent to distributed system.

10-7

Longitudinal Vibration



$\xi = x/l$

$\varphi_1 = (1-\xi)$

$\varphi_2 = \xi$

$u = u_1(1-\xi) + u_2\xi$

$$m_{ij} = \int \varphi_i \varphi_j \, dm \qquad r = R_1(1 - \tfrac{1}{2}\xi)$$

$$dm = \rho\, 2\pi r t \cdot dx \quad , \quad \rho = \text{mass density}$$

$$A = 2\pi r t$$

$$\therefore m_{11} = \rho\, 2\pi t R_1 \int_0^1 (1-\tfrac{1}{2}\xi)(1-\xi)^2 l \, d\xi = m_1 \int_0^1 \left(1 - \tfrac{5}{2}\xi + 2\xi^2 - \tfrac{1}{2}\xi^3\right) d\xi$$

where $m_1 = \rho 2\pi t R_1 l = \rho A_1 l$

$$m_{11} = m_1\left(1 - \tfrac{5}{4} + \tfrac{2}{3} - \tfrac{1}{8}\right) = \frac{m_1}{24}(24 - 30 + 16 - 3) = \frac{7}{24} m_1$$

$A_1 = 2\pi t R_1$

10-7 Cont.

$$m_{12} = m_1 \int_0^1 (1-\tfrac{1}{2}\xi)(1-\xi)\xi \, d\xi = \frac{3}{24} m_1$$

$$m_{22} = m_1 \int_0^1 (1-\tfrac{1}{2}\xi)\xi^2 \, d\xi = \frac{5}{24} m_1$$

$$\therefore \; m = \frac{m_1}{24}\begin{bmatrix} 7 & 3 \\ 3 & 5 \end{bmatrix}$$

$$k_{ij} = \int EA \varphi_i' \varphi_j' \, \ell \, d\xi \qquad \varphi_1' = \frac{d\varphi_1}{dx} = -\frac{1}{\ell}, \qquad \varphi_2' = \frac{1}{\ell}$$

$$k_{11} = E\frac{2\pi t R_1}{\ell^2}\int_0^1 (1-\tfrac{1}{2}\xi)\,\ell\, d\xi = \frac{EA_1}{\ell}(1-\tfrac{1}{4}) = \frac{3}{4}\frac{EA_1}{\ell}$$

where $A_1 = 2\pi t R_1$

$$k_{12} = -\frac{3}{4}\frac{EA_1}{\ell} = k_{21}, \qquad k_{22} = \frac{3}{4}\frac{EA_1}{\ell}$$

$$\therefore \; k = \frac{3}{4}\frac{EA_1}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Eq. of motion

$$\frac{\rho A_1 \ell}{24}\begin{bmatrix} 7 & 3 \\ 3 & 5 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \frac{3}{4}\frac{EA_1}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Let $\lambda = \dfrac{\omega^2 \rho \ell^2}{18E}$

$$\begin{vmatrix} (1-7\lambda) & -(1+3\lambda) \\ -(1+3\lambda) & (1-5\lambda) \end{vmatrix} = 0 \quad \therefore \; \lambda(26\lambda - 6) = 0$$

$$\lambda_0 = 0$$

$$\lambda_1 = \frac{6}{26} = \frac{\omega^2 \rho \ell^2}{18E} = \frac{\omega^2 (\rho A_1 \ell)\ell}{18EA_1} = \frac{\omega^2 m_1 \ell}{18EA_1}$$

10-7 Cont.

$$\omega_1^2 = \left(\frac{6}{26}\times 18\right)\left(\frac{E}{\rho\ell^2}\right) = 4.1538\left(\frac{E}{\rho\ell^2}\right)$$

$$\omega_1 = 2.038\sqrt{\frac{EA_1}{m_1\ell}} = \underline{\underline{2.038\sqrt{\frac{E}{\rho\ell^2}}}}$$ for 1-element solution for Longit. vib.

where A_1 = cross section area at ①

$M_1 = m_1\ell$ = mass of entire uniform cylinder of radius R_1 and length ℓ.

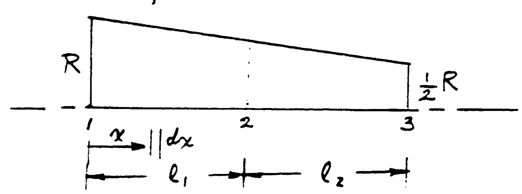
If we have a uniform cylinderical shell of Radius R_1 and length ℓ, with one end fixed with other end free, treated like a helical spring with effective mass $\frac{m_1}{3}$ at free end, the natural frequency will be

$$\omega_1 = \sqrt{\frac{EA_1}{\left(\frac{m_1}{3}\right)\ell}} = 1.732\sqrt{\frac{EA_1}{m_1\ell}}$$

10-8 Same conical tube as in 10-7 treated as two elements of equal length, 1-2 and 2-3.

$$r_{1-2} = R\left(1-\frac{1}{4}\frac{x}{\ell_1}\right)$$

$$\xi = \frac{x}{\ell_1}$$



$$m_{ij} = \int \varphi_i\varphi_j\, dm = \int \varphi_i\varphi_j\, 2\pi\rho t\, r_{1-2}\, dx = 2\pi\rho t R\int_0^1 (1-\tfrac{1}{4}\xi)\varphi_i\varphi_j \ell_1 d\xi$$

$$m_{11} = 2\pi\rho t R\ell_1\int_0^1 (1-\tfrac{1}{4}\xi)(1-\xi)^2 d\xi = 2\pi\rho t R\ell_1(.3125)$$

$$m_{12} = \text{"}\int_0^1 (1-\tfrac{1}{4}\xi)(1-\xi)\xi\, d\xi = \text{"}\ (.1458)$$

$$m_{22} = \text{"}\int_0^1 (1-\tfrac{1}{4}\xi)\xi^2 d\xi = \text{"}\ (.2708)$$

10-8 Cont.

$$m = 2\pi\rho t R \ell_1 \begin{bmatrix} .3125 & .1458 \\ .1458 & .2708 \end{bmatrix}$$

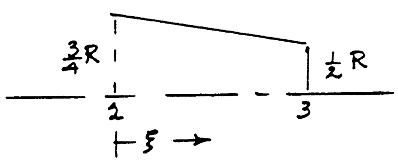
$$k_{ij} = \int_0^1 EA\varphi_i'\varphi_j'\ell_1\, d\xi \qquad \therefore\ k_{11} = \int_0^1 2\pi RE\left(1-\tfrac{1}{4}\xi\right)\left(-\tfrac{1}{\ell_1}\right)^2 \ell_1\, d\xi$$

$$= \frac{2\pi RE}{\ell_1}(.8750)$$

$$k_{12} = -k_{11}, \qquad k_{22} = k_{11}$$

$$k = \frac{2\pi RE}{\ell_1}\begin{bmatrix} .8750 & -.8750 \\ -.8750 & .8750 \end{bmatrix}$$

Element 2-3



$$r = \tfrac{3}{4}R\left(1-\tfrac{1}{3}\xi\right)$$

$$A = 2\pi t r = 2\pi t R\tfrac{3}{4}\left(1-\tfrac{1}{3}\xi\right)$$

$$dm = 2\pi\rho t r\, dx = 2\pi\rho t R\left(1-\tfrac{1}{3}\xi\right)\ell_1\, d\xi$$

$$m_{11} = \tfrac{3}{2}\pi\rho t R\ell_1\int_0^1\left(1-\tfrac{1}{3}\xi\right)(1-\xi)^2 d\xi = \tfrac{3}{2}\pi\rho t R\ell_1\,(.3056)$$

$$m_{12} = \tfrac{3}{2}\pi\rho t R\ell_1\,(.1389), \qquad m_{22} = \tfrac{3}{2}\pi\rho t R\ell_1\,(.2500)$$

$$m = \tfrac{3}{2}\pi\rho t R\ell_1\begin{bmatrix} .3056 & .1389 \\ .1389 & .250 \end{bmatrix} = 2\pi\rho t R\ell_1\begin{bmatrix} .2292 & .1042 \\ .1042 & .1875 \end{bmatrix}$$

$$k_{11} = \tfrac{3}{2}\pi\rho\frac{tR}{\ell_1}\int_0^1\left(1-\tfrac{1}{3}\xi\right)d\xi = \tfrac{3}{2}\pi\rho\frac{tR}{\ell_1}\left(\tfrac{5}{6}\right)$$

$$k_{12} = -k_{11}, \qquad k_{22} = k_{11}$$

$$\therefore\ k = \tfrac{3}{2}\pi\times\tfrac{5}{6}\,\rho\frac{tR}{\ell_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

10-8 Cont. Assemble element matrices

$$m = 2\pi\rho t R \ell_1 \begin{bmatrix} .3125 & .1458 & 0 \\ .1458 & (.2708+.2292) & .1042 \\ 0 & .1042 & .1875 \end{bmatrix}$$

$$k = \frac{\pi R E t}{\ell_1} \begin{bmatrix} 1.75 & -1.75 & \\ -1.75 & (1.75+1.25) & -1.25 \\ & -1.25 & 1.25 \end{bmatrix} \qquad \lambda = \frac{\omega^2 \rho \ell_1}{E}$$

With $u_1 = 0$, eqs. of motion **leads to** chara. eq.

$$\left| -\lambda \begin{bmatrix} .5000 & .1042 \\ .1042 & .1875 \end{bmatrix} + \begin{bmatrix} 3.00 & -1.25 \\ -1.25 & 1.25 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} (3.00 - .500\lambda) & -(1.25 + .1042\lambda) \\ -(1.25 + .1042\lambda) & (1.25 - .1875\lambda) \end{vmatrix} = 0$$

$$\lambda^2 - 17.46\lambda + 26.380 = 0$$

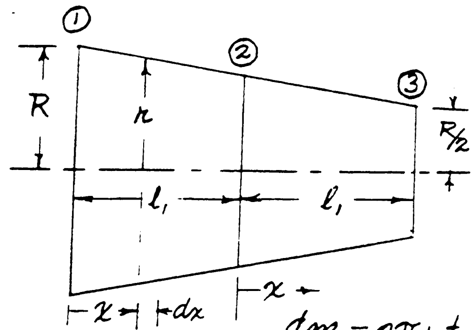
$$\lambda_1 = 1.6708 \qquad \omega_1 = 1.2926\sqrt{\frac{E}{\rho \ell_1^2}}$$

$$\lambda_2 = 15.7892 \qquad \omega_2 = 3.9736\sqrt{\frac{E}{\rho \ell_1^2}} \qquad \ell_1 = \frac{\ell}{2}$$

$$\therefore \quad \omega_1 = 2.585\sqrt{\frac{E}{\rho \ell^2}}, \qquad \omega_2 = 7.947\sqrt{\frac{E}{\rho \ell^2}}$$

longitudinal vibr (2 elements)

10 – 9(a) Torsional Vibration of Conical Shell



$$r_{1-2} = R(1-\tfrac{1}{4}\xi)$$

$$\xi = \frac{x}{\ell_1}, \quad dx = \ell_1 d\xi$$

$$r_{2-3} = \tfrac{3}{4}R(1-\tfrac{1}{3}\xi)$$

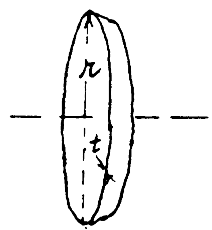
$$\varphi_1 = (1-\xi)$$

$$\varphi_2 = \xi$$

$$\varphi_1' = -\frac{1}{\ell_1}$$

$$\varphi_2' = \frac{1}{\ell_1}$$

$$dm = 2\pi r t \rho\, dx$$



$$dJ_{1-2} = r^2 dm = 2\pi\rho t R^3(1-\tfrac{1}{4}\xi)^3 \ell_1 d\xi$$

$I_p = 2\pi r t \cdot r^2 = 2^{nd}$ moment (polar) of cross-section area

Stiffness of element 1-2

$$k_{ij} = G\int_0^1 I_p \varphi_i' \varphi_j'\, dx = G\cdot 2\pi t\, R^3 \int_0^1 (1-\tfrac{1}{4}\xi)^3 \varphi_i'\varphi_j' \ell_1 d\xi$$

$$k_{11} = G\cdot 2\pi t\, R^3 \int_0^1 (1-\tfrac{1}{4}\xi)^3\left(\frac{1}{\ell_1^2}\right)\ell_1 d\xi = \frac{G2\pi t R^3}{\ell_1}\int_0^1 (1-\tfrac{3}{4}\xi+\tfrac{3}{16}\xi^2-\tfrac{1}{64}\xi^3)d\xi$$

$$= \frac{G2\pi t R^3}{\ell_1}\left(1-\frac{3}{8}+\frac{1}{16}-\frac{1}{256}\right) = \frac{G2\pi t R^3}{\ell_1}(.6836)$$

$$k_{12} = k_{21} = -k_{11}, \qquad k_{22} = k_{11}$$

$$\therefore \quad K_{1-2} = \frac{G2\pi t R^3}{\ell_1}(.6836)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Stiffness of element 2-3

$$k_{11} = \frac{G2\pi t}{\ell_1}(.75R)^3\int_0^1 (1-\tfrac{1}{3}\xi)^3 d\xi = \frac{G2\pi t R^3}{\ell_1}(.4219)\int_0^1 (1-\xi+\tfrac{1}{3}\xi^2-\tfrac{1}{27}\xi^3)d\xi$$

$$= \frac{G2\pi t R^3}{\ell_1}(.4219)(.6019) = \frac{G2\pi t R^3}{\ell_1}(.2539)$$

$$K_{2-3} = \frac{G2\pi t R^3}{\ell_1}(.2539)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{G2\pi t R^3}{\ell_1}(.6836)\begin{bmatrix} .3714 & -.3714 \\ -.3714 & .3714 \end{bmatrix}$$

<u>10 − 9 (a) Cont.</u>

$$\text{Combined stiffness} = \frac{G2\pi tR^3}{\ell/2}(.6836)\begin{bmatrix} 1 & -1 & \\ -1 & 1.3714 & -.3714 \\ & -.3714 & .3714 \end{bmatrix}$$

For $\theta_1 = 0$

$$K\theta = (1.3672)\frac{G2\pi tR^3}{\ell}\begin{bmatrix} 1.3714 & -.3714 \\ -.3714 & .3714 \end{bmatrix}\begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix}$$

<u>Mass matrix, element 1-2</u>

$$J_{ij} = \int_0^1 \varphi_i \varphi_j \, dJ = 2\pi\rho t R^3 \int_0^1 (1-\tfrac{1}{4}\xi)^3 \varphi_i \varphi_j \, \ell_1 \, d\xi$$

$$J_{11} = 2\pi\rho t R^3 \int_0^1 (1-\tfrac{1}{4}\xi)^3 (1-\xi)^2 \ell_1 \, d\xi$$

$$= 2\pi\rho t R^3 \ell_1 \int_0^1 \left(1 - \tfrac{11}{4}\xi + \tfrac{43}{16}\xi^2 - \tfrac{73}{64}\xi^3 + \tfrac{7}{32}\xi^4 - \tfrac{1}{64}\xi^5\right) d\xi$$

$$= \quad '' \quad \left(1 - \tfrac{11}{8} + \tfrac{43}{48} - \tfrac{73}{256} + \tfrac{7}{160} - \tfrac{1}{384}\right)$$

$$= 2\pi\rho t R^3 \ell_1 (.2768)$$

$$J_{12} = 2\pi\rho t R^3 \ell_1 \int_0^1 (1-\tfrac{1}{4}\xi)^3 (1-\xi)\xi \, d\xi$$

$$'' \quad \int_0^1 \left(\xi - \tfrac{7}{4}\xi^2 + \tfrac{15}{16}\xi^3 - \tfrac{13}{64}\xi^4 + \tfrac{1}{64}\xi^5\right) d\xi$$

$$= 2\pi\rho t R^3 \ell_1 (.1130)$$

$$J_{22} = 2\pi\rho t R^3 \ell_1 \int_0^1 (1-\tfrac{1}{4}\xi)^3 \xi^2 \, d\xi = \int_0^1 \left(\xi^2 - \tfrac{3}{4}\xi^3 + \tfrac{3}{16}\xi^4 - \tfrac{1}{64}\xi^5\right) d\xi \, (\quad)$$

$$= 2\pi\rho t R^3 \ell_1 \left(\tfrac{1}{3} - \tfrac{3}{16} + \tfrac{3}{80} - \tfrac{1}{384}\right) = 2\pi\rho t R^3 \ell_1 (.1804)$$

10 – 9(a) Cont.

$$J_{1-2} = 2\pi\rho t R^3 \ell_1 \begin{bmatrix} .2768 & .1130 \\ .1130 & .1804 \end{bmatrix} \qquad (\ell_1 = \tfrac{\ell}{2})$$

Mass matrix, element 2 – 3 $\qquad r_{2-3} = \frac{3}{4}R(1 - \frac{1}{3}\xi)$

$$dJ_{2-3} = 2\pi r t\rho \cdot r^2 = 2\pi\rho t(\tfrac{3}{4}R)^3(1-\tfrac{1}{3}\xi)^3$$

$$J_{22} = 2\pi\rho t R^3(.4219)\int_0^1 (1-\tfrac{1}{3}\xi)^3(1-\xi)^2 \ell_1\, d\xi$$

$$\text{''} \quad \ell_1\int_0^1 (1 - 3\xi + \tfrac{10}{3}\xi^2 - 1.7036\xi^3 + .40737\xi^4 - .037043\xi^5)\, d\xi$$

$$= 2\pi\rho t R^3 \ell_1(.4219)(.2605) = 2\pi\rho t R^3 \ell_1 (.1099)$$

$$J_{23} = \text{same term} \times \ell_1\int_0^1 (1-\tfrac{1}{3}\xi)^3(1-\xi)\xi\, d\xi$$

$$= \quad \text{''} \quad \int_0^1 (\xi - 2\xi^2 + \tfrac{4}{3}\xi^3 - \tfrac{10}{27}\xi^4 + \tfrac{1}{27}\xi^5)\, d\xi$$

$$= 2\pi\rho t R^3 \ell_1 (.4219)(.0988) = 2\pi\rho t R^3 \ell_1 (.0417)$$

$$J_{33} = \text{same term} \times \ell_1\int_0^1 (1-\tfrac{1}{3}\xi)^3\xi^2\, d\xi = 2\pi\rho t R^3 \ell_1 (.4219)(.1438)$$

$$= 2\pi\rho t R^3 \ell_1 (.0607)$$

Combined mass term with $\theta_1 = 0$

$$2\pi\rho t R^3 \ell_1 \begin{bmatrix} (.1099 + .1804) & .0417 \\ .0417 & .0607 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} \qquad (\ell_1 = \tfrac{\ell}{2})$$

$$J = \pi\rho t R^3 \ell \begin{bmatrix} .2903 & .0417 \\ .0417 & .0607 \end{bmatrix}$$

10-9 (a) Cont.

Eq. of motion: Let $\lambda = \dfrac{\omega^2 \cancel{\pi} \rho \cancel{t} \cancel{R^3} \ell}{1.3672\, G\, 2\cancel{\pi} \cancel{t} \cancel{R^3}/\ell}$

$$\lambda = \frac{\omega^2 \rho \ell^2}{2.734\, G}$$

$$\left[-\lambda \begin{bmatrix} .2903 & .0417 \\ .0417 & .0607 \end{bmatrix} + \begin{bmatrix} 1.3714 & -.3714 \\ -.3714 & .3714 \end{bmatrix}\right] \begin{Bmatrix} \Theta_2 \\ \Theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{vmatrix} (1.3714 - .2903\lambda) & -(.3714 + .0417\lambda) \\ -(.3714 - .0417\lambda) & (.3714 - .0607\lambda) \end{vmatrix} = 0$$

$$\lambda^2 - 10.069\lambda + 23.358 = 0$$

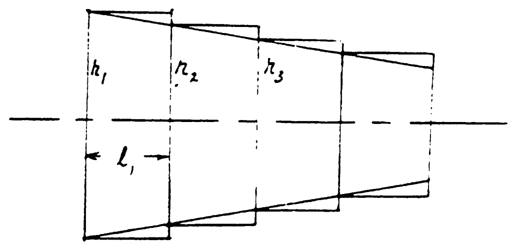
$\lambda_1 = 3.624$ $\qquad$ $\lambda_2 = 6.438$

$\omega_1^2 = 9.908 \dfrac{G}{\rho \ell^2}$ $\qquad$ $\omega_2^2 = 17.603 \dfrac{G}{\rho \ell^2}$

$\underline{\underline{\omega_1 = 3.147 \sqrt{\dfrac{G}{\rho \ell^2}}}}$ $\qquad$ $\underline{\underline{\omega_2 = 4.196 \sqrt{\dfrac{G}{\rho \ell^2}}}}$

Results unchecked.

10 – 9 (b)



Torsion problem.

Assume short elements ∴ constant radius for each section.

Element stiffness:

$$A_i = 2\pi r_i t\,, \quad I_{p_i} = A r_i^2\,, \quad GI_{p_i} = GA_i r^2$$

$$K_i = \frac{GI_{p_i}}{l_i} = \frac{G\,2\pi t\, r_i^3}{l_i}$$

Assume straight line variation in twist ∴ φ_{1_i} , φ_{2_i} applies to each element $\varphi_{1_i} = (1-\xi)$, $\varphi_{2_i} = \xi$ where $\xi = \frac{x}{l_i}$. Then $K_i = \frac{GI_{p_i}}{l_i}\int_0^1 \varphi'_{1_i}\varphi'_{2_i} l_i\, d\xi$

$$\int_0^1 \varphi'_i \varphi'_j l_i\, d\xi = \pm 1$$

$$K \text{ for element} = G\left(\frac{2\pi t r_i^3}{l_i}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad l_i = \frac{l}{n}$$

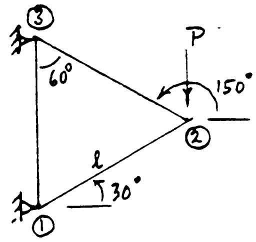
Element mass:

$$J \text{ for element} = \frac{2\pi t \rho l_i r_i^3}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

N section eq.

$$\rho\,\frac{2\pi t l}{6n}\begin{bmatrix} 2r_1^3 & r_1^3 & & & \\ r_1^3 & 2(r_1^3+r_2^3) & r_2^3 & & \\ & r_2^3 & 2(r_2^3+r_3^3) & r_3^3 & \\ & & r_3^3 & 2(r_3^3+r_4^3) & \\ & & & & \text{etc.} \end{bmatrix}\begin{Bmatrix} \ddot\theta_1 \\ \ddot\theta_2 \\ \ddot\theta_3 \\ \ddot\theta_4 \\ \vdots \end{Bmatrix} + \frac{G\,2\pi t n}{l}\begin{bmatrix} r_1^3 & -r_1^3 & \\ -r_1^3 & (r_1^3+r_2^3) & -r_2^3 \\ & -r_2^3 & (r_2^3+r_3^3) \\ & & \text{etc} \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \vdots \end{Bmatrix}$$

10 - 10



Element 1-2 $\alpha = 30^\circ$ $C = .866,$ $S = .500$

$c^2 = .750$ $s^2 = .250$ $cs = .433$

$$\bar{k}_{12} = \frac{EA}{\ell}\left[\begin{array}{cc|cc} \cdot & \cdot & & \\ & & .750 & .433 \\ & & .433 & .250 \end{array}\right] \begin{Bmatrix} \bar{u}_1 = 0 \\ \bar{v}_1 = 0 \\ \bar{u}_2 \\ \bar{v}_2 \end{Bmatrix}$$

Element 2-3, $\alpha = 150^\circ$ $C = -.866$ $S = .50$

$$\bar{k}_{23} = \frac{EA}{\ell}\left[\begin{array}{cc|cc} .750 & -.433 & -.750 & .433 \\ -.433 & .250 & .433 & -.250 \\ \hline -.750 & .433 & .750 & -.433 \\ .433 & -.250 & -.433 & .250 \end{array}\right] \begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ \bar{u}_3 \\ \bar{v}_3 \end{Bmatrix}$$

Assemble

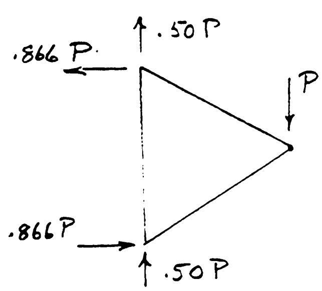
$$\begin{Bmatrix} F_{2x} = 0 \\ F_{2y} = -P \\ F_{3x} \\ F_{3y} \end{Bmatrix} = \frac{EA}{\ell}\left[\begin{array}{cc|cc} 1.50 & 0 & -.75 & .433 \\ 0 & .50 & .433 & -.25 \\ \hline -.75 & .433 & .75 & -.433 \\ .433 & -.25 & -.433 & .25 \end{array}\right] \begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ \bar{u}_3 = 0 \\ \bar{v}_3 = 0 \end{Bmatrix}$$

$$F_{2x} = 0 = \frac{EA}{\ell}\, 1.50\, \bar{u}_2 \qquad \therefore\ \bar{u}_2 = 0$$

$$F_{2y} = -P = \frac{EA}{\ell}\, .50\, \bar{v}_2 \qquad \therefore\ \bar{v}_2 = -\frac{2P\ell}{EA}$$

$$F_{3x} = \frac{EA}{\ell}\, .433\, \bar{v}_2 = \frac{EA}{\ell}\, .433\left(\frac{-2P\ell}{EA}\right) = -.866P$$

$$F_{3y} = -.25\, \bar{v}_2 \frac{EA}{\ell} = -.25 \frac{EA}{\ell}\left(-\frac{2P\ell}{EA}\right) = 0.50P$$



10 − 11 See Example 10.4-1 for the 6 × 6 general matrix equation which applies to this problem. With zero displacements for u_1, v_2, u_3 and v_3 the equation reduces to

$$\begin{Bmatrix} F_{1x} \\ F_{1y}=0 \\ F_{2x}=P \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix} = \left(\frac{EA}{25\ell}\right) \begin{bmatrix} \cdots & 12 & -16 & \cdots \\ \cdots & 50.67 & -12 & \cdots \\ \cdots & -12 & 47.25 & \cdots \\ \cdots & -9 & 12 & \cdots \\ \cdots & 0 & -31.25 & \cdots \\ \cdots & -41.67 & 0 & \cdots \end{bmatrix} \begin{Bmatrix} u_1=0 \\ v_1 \\ u_2 \\ v_2=0 \\ u_3=0 \\ v_3=0 \end{Bmatrix}$$

columns crossed out

The displacements v_1 and u_2 are available from the second and third equations

$$0 = 50.67\, v_1 - 12\, u_2 \qquad \therefore\; v_1 = .2368\, u_2$$

$$P = \left(\frac{EA}{25\ell}\right)\left[-12\, v_1 + 47.25\, u_2\right] = \left(\frac{EA}{25\ell}\right) 44.41\, u_2$$

$$\therefore\; \underline{\underline{u_2 = \frac{P\ell}{EA}(.5629)}}\,, \qquad \underline{\underline{v_1 = (.1333)\frac{P\ell}{EA}}}$$

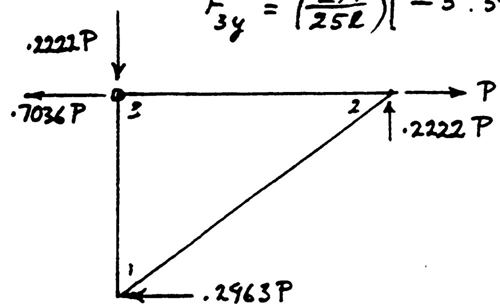
The reaction forces are then available from the 4 remaining equations:

$$F_{1x} = \left(\frac{EA}{25\ell}\right)\left[1.5996 - 9.0064\right]\frac{P\ell}{EA} = -.2963\,P$$

$$F_{2y} = \left(\frac{EA}{25\ell}\right)\left[-1.1997 + 6.7548\right]\frac{P\ell}{EA} = 0.2222\,P$$

$$F_{3x} = \left(\frac{EA}{25\ell}\right)\left[-17.5906\right]\frac{P\ell}{EA} = -.7036\,P$$

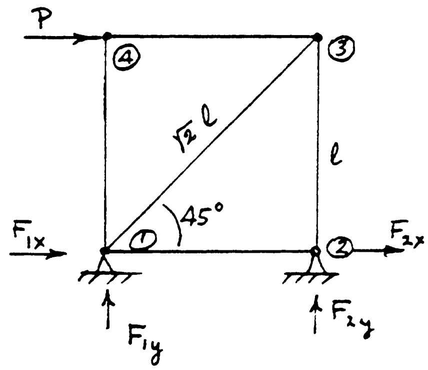
$$F_{3y} = \left(\frac{EA}{25\ell}\right)\left[-5.546\right]\frac{P\ell}{EA} = -.2222\,P$$



Check $\Sigma M_3 = 0$

$$\Sigma F_x = P - F_{1x} - F_{3x}$$
$$= P - .2963\,P - .7036\,P = 0$$

10 — 12



$$\bar{k}_{12} = \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \bar{m}_{12} = \frac{ml}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Element 1-2, $\alpha = 0$, $s = 0$, $c = 1$

$$\bar{k}_{1-2} = \frac{EA}{l}\begin{array}{c} \left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} 1 \\ \\ 2 \\ \\ \end{array}$$

Element 2-3 $\alpha = 90°$, $s = 1$, $c = 0$

$$\bar{k}_{2-3} = \frac{EA}{l}\begin{array}{c} \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right] \end{array} \begin{array}{c} 2 \\ \\ 3 \\ \\ \end{array}$$

Element 3-4 $\alpha = 180°$ $s = 0$, $c = -1$

$$\bar{k}_{3-4} = \frac{EA}{l}\begin{array}{c} \left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} 3 \\ \\ 4 \\ \\ \end{array}$$

Element 4-1 $\alpha = 270°$ $s = -1$, $c = 0$

$$\bar{k}_{41} = \frac{EA}{l}\begin{array}{c} \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right] \end{array} \begin{array}{c} 4 \\ \\ 1 \\ \\ \end{array}$$

Element 1 - 3 $\alpha = 45°$, $s = \frac{\sqrt{2}}{2}$, $c = \frac{\sqrt{2}}{2}$ Length $= \sqrt{2}\,l$

$$\bar{k}_{13} = \frac{EA}{\sqrt{2}\,l}\begin{array}{c} \left[\begin{array}{cc|cc} .5 & .5 & -.5 & -.5 \\ .5 & .5 & -.5 & -.5 \\ \hline -.5 & -.5 & .5 & .5 \\ -.5 & -.5 & .5 & .5 \end{array}\right] \end{array} \begin{array}{c} 1 \\ \\ 3 \\ \\ \end{array}$$

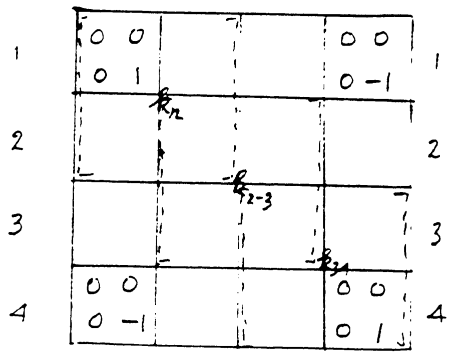
10 – 12 Cont.

$$\begin{Bmatrix} \bar{F}_{1x} \\ \bar{F}_{1y} \\ \hline \bar{F}_{2x} \\ \bar{F}_{2y} \\ \hline \bar{F}_{3x} \\ \bar{F}_{3y} \\ \hline \bar{F}_{4x} \\ \bar{F}_{4y} \end{Bmatrix} = \left[\cdot\cdot K \cdot\cdot \right] \begin{Bmatrix} \bar{u}_1 = 0 \\ \bar{v}_1 = 0 \\ \hline \bar{u}_2 = 0 \\ \bar{v}_2 = 0 \\ \hline \bar{u}_3 \\ \bar{v}_3 \\ \hline \bar{u}_4 \\ \bar{v}_4 \end{Bmatrix}$$

Since all displacements for ① and ② are zero, we need only the lower part of this matrix or the lower right section of K. However for instructional purposes we will fill in the elements into the 8 x 8 space for K.

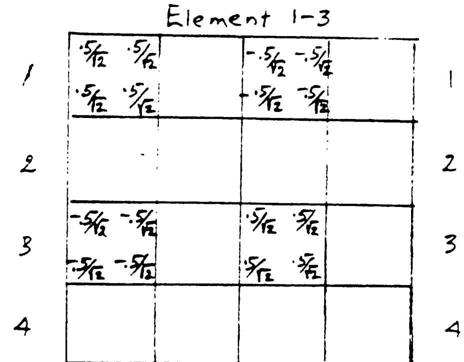
There is no problem with placement of $\bar{k}_{12}$, $\bar{k}_{23}$ and $\bar{k}_{34}$. Element $\bar{k}_{4-1}$ and $\bar{k}_{1-3}$ must be placed as follows:

Element 4-1



dotted squares indicate placement of k_{12}, k_{23}, k_{34}

Element 1-3



Superimpose this onto diagram on left.

Considering only those terms in lower right quarter, we have

$$\begin{Bmatrix} \bar{F}_{3x} = 0 \\ \bar{F}_{3y} = 0 \\ \bar{F}_{4x} = P \\ \bar{F}_{4y} = 0 \end{Bmatrix} = \frac{EA}{\ell} \begin{bmatrix} (1 + .5/\sqrt{2}) & .5/\sqrt{2} & -1 & 0 \\ .5/\sqrt{2} & (1 + .5/\sqrt{2}) & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \bar{u}_3 \\ \bar{v}_3 \\ \bar{u}_4 \\ \bar{v}_4 \end{Bmatrix}$$

10 – 12 Cont. The last matrix give

$$(1) \qquad 0 = \frac{EA}{\ell}\left[(1+\tfrac{.5}{\sqrt2})\bar{u}_3 + \tfrac{.5}{\sqrt2}\bar{v}_3 - \bar{u}_4\right]$$

$$(2) \qquad 0 = \frac{EA}{\ell}\left[\tfrac{.5}{\sqrt2}\bar{u}_3 + (1+\tfrac{.5}{\sqrt2})\bar{v}_3\right] \rightarrow \quad \bar{u}_3 = -3.828\,\bar{v}_3$$

$$(3) \qquad P = \frac{EA}{\ell}\left[-\bar{u}_3 + \bar{u}_4\right]$$

$$(4) \qquad 0 = \frac{EA}{\ell}\left[\bar{v}_4\right] \qquad \therefore \ \bar{v}_4 = 0$$

add (1) and (3) to eliminate $\bar{u}_4$ & subst (2) into it.

$$\frac{P\ell}{EA} = \frac{.5}{\sqrt2}\bar{u}_3 + \frac{.5}{\sqrt2}\bar{v}_3 \quad \rightarrow \quad \bar{u}_3 + \bar{v}_3 = \frac{1}{.3536}\frac{P\ell}{AE}$$

$$\bar{u}_3 - \frac{1}{3.828}\bar{u}_3 = \frac{P\ell}{.3536\,AE} \qquad \therefore \ \underline{\underline{\bar{u}_3 = 3.828\frac{P\ell}{AE}}}$$

$$\underline{\underline{\bar{v}_3 = -\frac{P\ell}{AE}}}$$

$$\underline{\underline{\bar{u}_4 = 4.828\frac{P\ell}{AE}}}$$

Complete Equation

$$\begin{Bmatrix} \bar{F}_{1x} \\ \bar{F}_{1y} \\ \bar{F}_{2x} \\ \bar{F}_{2y} \\ \bar{F}_{3x} \\ \bar{F}_{3y} \\ \bar{F}_{4x} \\ \bar{F}_{4y} \end{Bmatrix} = \frac{EA}{\ell}\left[\begin{array}{cc:cc:cc:cc} (1+.5/\sqrt2) & .5/\sqrt2 & -1 & 0 & -.5/\sqrt2 & -.5/\sqrt2 & 0 & 0 \\ .5/\sqrt2 & (1+.5/\sqrt2) & 0 & 0 & -.5/\sqrt2 & -.5/\sqrt2 & 0 & -1 \\ \hdashline -1 & 0 & 1 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 1 & 0 & -1 & & \\ \hdashline -.5/\sqrt2 & -.5/\sqrt2 & 0 & 0 & (1+.5/\sqrt2) & .5/\sqrt2 & -1 & 0 \\ -.5/\sqrt2 & -.5/\sqrt2 & 0 & -1 & .5/\sqrt2 & (1+.5/\sqrt2) & 0 & 0 \\ \hdashline 0 & 0 & & & -1 & 0 & 1 & 0 \\ 0 & -1 & & & 0 & 0 & 0 & 1 \end{array}\right] \begin{Bmatrix} \bar{u}_1 = 0 \\ \bar{v}_1 = 0 \\ \bar{u}_2 = 0 \\ \bar{v}_2 = 0 \\ \bar{u}_3 \\ \bar{v}_3 \\ \bar{u}_4 \\ \bar{v}_4 \end{Bmatrix}$$

Mass matrix

$$\bar{m} = \frac{m\ell}{6}\begin{bmatrix} 2c^2 & 2cs & c^2 & -cs \\ 2cs & 2s^2 & cs & s^2 \\ c^2 & cs & 2c^2 & 2cs \\ cs & s^2 & 2cs & 2s^2 \end{bmatrix}$$

Element a: $\alpha = 0 \quad \cos(\alpha) = 1 \;,\; \sin(\alpha) = 0$

$$= \frac{m\ell}{6}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \ddot{\bar{u}}_1 \\ \ddot{\bar{v}}_1 \\ \ddot{\bar{u}}_2 \\ \ddot{\bar{v}}_2 \end{Bmatrix}$$

Element b: $\alpha = 90 \quad \cos(\alpha) = 0 \;,\; \sin(\alpha) = 1$

$$= \frac{m\ell}{6}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}\begin{Bmatrix} \ddot{\bar{u}}_2 \\ \ddot{\bar{v}}_2 \\ \ddot{\bar{u}}_3 \\ \ddot{\bar{v}}_3 \end{Bmatrix}$$

element D: $\alpha = 180 \quad \cos(\alpha) = -1 \quad \sin(\alpha) = 0$

$$= \frac{m\ell}{6}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \ddot{\bar{u}}_3 \\ \ddot{\bar{v}}_3 \\ \ddot{\bar{u}}_4 \\ \ddot{\bar{v}}_4 \end{Bmatrix}$$

10-12 CONT

Element E: $\alpha = 270 \quad \sin(270) = -1 \quad \cos(270) = 0$

$$= \frac{m\ell}{6}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_A \\ \ddot{v}_A \\ \ddot{u}_1 \\ \ddot{v}_1 \end{Bmatrix}$$

$c^2 = \frac{1}{2} = cs = s^2$

Element C: $\alpha = 225 \quad \cos(225) = -1/\sqrt{2} \quad \sin(\alpha) = -1/\sqrt{2}$

$$= \frac{m\ell\sqrt{2}}{6}\begin{bmatrix} 1 & 1 & 1/2 & 1/2 \\ 1 & 1 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1 & 1 \\ 1/2 & 1/2 & 1 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{u}_3 \\ \ddot{v}_3 \\ \ddot{u}_1 \\ \ddot{v}_1 \end{Bmatrix}$$

ADD Elements a+b

$$\frac{m\ell}{6}\begin{bmatrix} 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{v}_1 \\ \ddot{u}_2 \\ \ddot{v}_2 \\ \ddot{u}_3 \\ \ddot{v}_3 \end{Bmatrix}$$

10-12 cont

Elements a, b + D

$$\frac{ml}{6}\begin{bmatrix} 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{u}_2 \\ \bar{v}_2 \\ \bar{u}_3 \\ \bar{v}_3 \\ \bar{u}_4 \\ \bar{v}_4 \end{Bmatrix}$$

rewrite element E as

$$\frac{ml}{6}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{u}_2 \\ \bar{v}_2 \\ \bar{u}_3 \\ \bar{v}_3 \\ \bar{u}_4 \\ \bar{v}_4 \end{Bmatrix}$$

elemets a, b, D + E

$$\frac{ml}{6}\begin{bmatrix} 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{u}_2 \\ \bar{v}_2 \\ \bar{u}_3 \\ \bar{v}_3 \\ \bar{u}_4 \\ \bar{v}_4 \end{Bmatrix}$$

rewrite element C as:

$$\frac{m\ell}{6}\begin{bmatrix} \sqrt{2} & \sqrt{2} & 0 & 0 & \sqrt{2}/2 & \sqrt{2}/2 & 0 & 0 \\ \sqrt{2} & \sqrt{2} & 0 & 0 & \sqrt{2}/2 & \sqrt{2}/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 & 0 & \sqrt{2} & \sqrt{2} & 0 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 & 0 & \sqrt{2} & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{u}_2 \\ \bar{v}_2 \\ \bar{u}_3 \\ \bar{v}_3 \\ \bar{u}_4 \\ \bar{v}_4 \end{Bmatrix}$$

elements a, b, D, E & C

$$\frac{m\ell}{6}\begin{bmatrix} 2+2/\sqrt{2} & \sqrt{2} & 1 & 0 & 0 & 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2} & 2+2/\sqrt{2} & 0 & 0 & 0 & 0 & \sqrt{2}/2 & 1+\sqrt{2}/2 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 & 0 & 2+\sqrt{2} & \sqrt{2} & 1 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 & 1 & \sqrt{2} & 2+\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}\begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{u}_2 \\ \bar{v}_2 \\ \bar{u}_3 \\ \bar{v}_3 \\ \bar{u}_4 \\ \bar{v}_4 \end{Bmatrix}$$

→ Now impose zero displacement @ 1 & 2

$$\rightarrow \frac{m\ell}{6}\begin{bmatrix} 2+\sqrt{2} & \sqrt{2} & 1 & 0 \\ \sqrt{2} & 2+\sqrt{2} & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}\begin{Bmatrix} \bar{u}_3 \\ \bar{v}_3 \\ \bar{u}_4 \\ \bar{v}_4 \end{Bmatrix}$$

10-12 cont

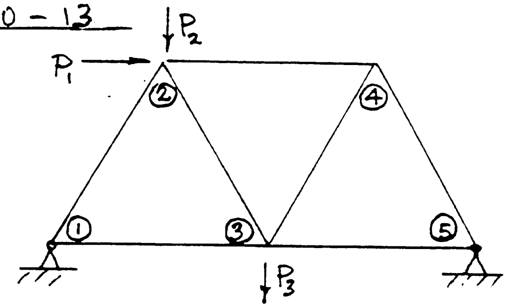
$$\begin{Bmatrix} F_{3x}=0 \\ F_{3y}=0 \\ F_{4x}=P \\ F_{4y}=0 \end{Bmatrix} = \frac{EA}{\ell}\begin{bmatrix} 1+.5/\sqrt{2} & .5/\sqrt{2} & -1 & 0 \\ .5/\sqrt{2} & 1+.5/\sqrt{2} & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} \bar{u}_3 \\ \bar{v}_3 \\ \bar{u}_4 \\ \bar{v}_4 \end{Bmatrix}$$

$$+ \frac{m\ell}{6}\begin{bmatrix} 2+\sqrt{2} & \sqrt{2} & 1 & 0 \\ \sqrt{2} & 2+\sqrt{2} & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}\begin{Bmatrix} \ddot{\bar{u}}_3 \\ \ddot{\bar{v}}_3 \\ \ddot{\bar{u}}_4 \\ \ddot{\bar{v}}_4 \end{Bmatrix}$$

Q = chol(m)
a = $Q^{-T} k Q^{-1}$

```
a =

    0.3964   -0.0666   -0.6018         0
   -0.0666    0.4571    0.3078         0
   -0.6018    0.3078    1.1350         0
         0         0         0    0.5000

EDU»eig(a)

ans =

    0.3978
    0.0360
    1.5548
    0.5000
```

10 - 13



Elements 1-2, 3-4, $\alpha = 60°$ $\quad c = .5$, $s = .866$, $cs = .433$

$c^2 = .250 \quad s^2 = .750$

$$\bar{k} = \frac{EA}{\ell}\left[\begin{array}{cc|cc} .250 & .433 & -.250 & -.433 \\ .433 & .750 & -.433 & -.750 \\ \hline & & .250 & .433 \\ & & & .750 \end{array}\right]$$

Elements 2-3, 4-5 $\quad \alpha = 300° \quad c = .5 \quad s = -.866 \quad cs = -.433$

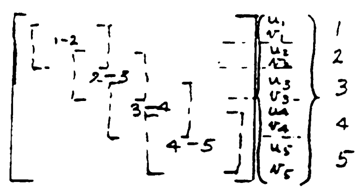
$c^2 = .250 \quad s^2 = .750$

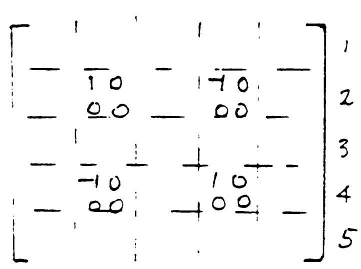
$$\bar{k} = \frac{EA}{\ell}\left[\begin{array}{cc|cc} .250 & -.433 & -.250 & .433 \\ & .750 & .433 & -.750 \\ & & .250 & -.433 \\ & & & .750 \end{array}\right]$$

Element 1-3, 2-4, 3-5 $\quad \alpha = 0 \quad c = 1 \quad s = 0 \quad cs = 0$

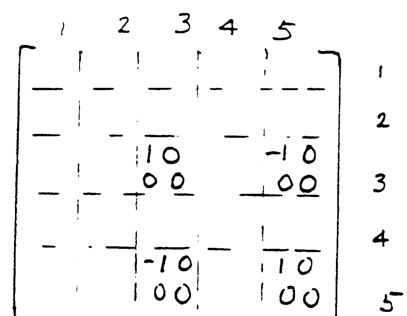
$c^2 = 1 \quad s^2 = 0$

$$\bar{k} = \frac{EA}{\ell}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



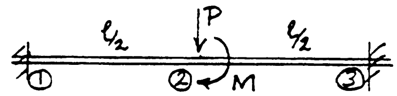


Element 2-4



Element 3-5

10 - 14



From Example 10.5-1 and noting that the ends 1 and 3 are fixed, the stiffness matrix and displacement vectors are:

$$\begin{Bmatrix} F_2 \\ M_2 \\ F_3 \\ M_3 \end{Bmatrix} = \frac{8EI}{\ell^3}\left[\begin{array}{cc|cc} 24 & 0 & -12 & 3\ell \\ 0 & 2\ell^2 & -3\ell & .5\ell^2 \\ \hline -12 & -3\ell & 12 & -3\ell \\ 3\ell & .5\ell^2 & -3\ell & \ell^2 \end{array}\right]\begin{Bmatrix} \bar{v}_2 \\ \theta_2 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} F_2 \\ M_2 \end{Bmatrix} = \begin{Bmatrix} -P \\ -M \end{Bmatrix} = \frac{8EI}{\ell^3}\begin{bmatrix} 24 & 0 \\ 0 & 2\ell^2 \end{bmatrix}\begin{Bmatrix} \bar{v}_2 \\ \theta_2 \end{Bmatrix}$$

$$\therefore \quad \bar{v}_2 = -\frac{P\ell^3}{192\,EI} \qquad \theta_2 = -\frac{M\ell}{16\,EI}$$

$$M_3 = \frac{8EI}{\ell^3}\left[3\ell\,\bar{v}_2 + .5\ell^2\theta_2\right] = \frac{8EI}{\ell^3}\left[3\ell\left(\frac{-P\ell^3}{192EI}\right) + .5\ell^2\left(\frac{-M\ell}{16EI}\right)\right]$$

---

10 - 15

Refer to Example 10.5-1 with $v_3 = \theta_3 = 0$

$$\left[-\frac{\omega^2 m\ell}{840}\begin{bmatrix} 312 & 0 \\ 0 & 2\ell^2 \end{bmatrix} + \frac{8EI}{\ell^3}\begin{bmatrix} 24 & 0 \\ 0 & 2\ell^2 \end{bmatrix}\right]\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Let $\lambda = \left(\frac{\omega^2 m\ell}{840}\right)\left(\frac{\ell^3}{8EI}\right) = \frac{m\ell^4\omega^2}{6720\,EI}$

$$\begin{vmatrix} (24 - 312\lambda) & 0 \\ 0 & 2\ell^2(1-\lambda) \end{vmatrix} = 0 \qquad \lambda_1 = \frac{24}{312} = \frac{m\ell^4\omega_1^2}{6720\,EI}$$

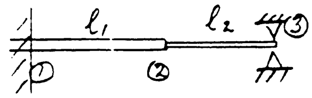
$$\lambda_2 = 1 = \frac{m\ell^4\omega_2^2}{6720\,EI}$$

$\therefore \quad \omega_1 = 22.74\sqrt{\frac{EI}{m\ell^4}} \quad \longrightarrow \quad 1.6\,\%$ high

$\omega_2 = 81.98\sqrt{\frac{EI}{m\ell^4}} \quad \longrightarrow \quad 33\,\%$ high.

Exact values $\begin{cases} \omega_1 = 22.37\sqrt{} \\ \omega_2 = 61.67\sqrt{} \end{cases}$

10 - 16



Element 1-2

$$\bar{k} = \frac{EI_1}{\ell_1^3}\begin{bmatrix} \cancel{R} & \cancel{0} & \cancel{0} \\ 0 & 12 & -6\ell_1 \\ 0 & -6\ell_1 & 4\ell_1^2 \end{bmatrix}\begin{Bmatrix} \cancel{\bar{u}_2} \\ \bar{v}_2 \\ \theta_2 \end{Bmatrix} \qquad \bar{m} = \frac{m_1\ell_1}{420}\begin{bmatrix} 156 & -22\ell_1 \\ -22\ell_1 & 4\ell_1^2 \end{bmatrix}\begin{Bmatrix} \ddot{\bar{v}}_2 \\ \ddot{\theta}_2 \end{Bmatrix}$$

Element 2-3

$$\bar{k} = \frac{EI_2}{\ell_2^3}\left[\begin{array}{ccc|ccc} R & 0 & 0 & -R & 0 & 0 \\ 0 & 12 & 6\ell_2 & 0 & -12 & 6\ell_2 \\ 0 & 6\ell_2 & 4\ell_2^2 & 0 & -6\ell_2 & 2\ell_2^2 \\ \hline -R & 0 & 0 & R & 0 & 0 \\ 0 & -12 & -6\ell_2 & 0 & 12 & -6\ell_2 \\ 0 & 6\ell_2 & 2\ell_2^2 & 0 & -6\ell_2 & 4\ell_2^2 \end{array}\right]\begin{Bmatrix} \bar{u}_2 = 0 \\ \bar{v}_2 \\ \theta_2 \\ \bar{u}_3 \\ 0 \\ \theta_3 \end{Bmatrix}$$

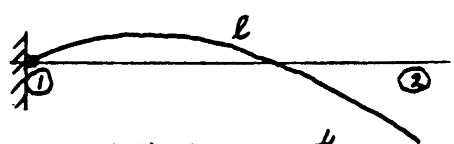
$$\bar{m} = \frac{m_2\ell_2}{420}\left[\begin{array}{ccc|ccc} N & 0 & 0 & \frac{1}{2}N & 0 & 0 \\ 0 & 156 & 22\ell_2 & 0 & 54 & -13\ell_2 \\ 0 & 22\ell_2 & 4\ell_2^2 & 0 & 13\ell_2 & -3\ell_2^2 \\ \hline \frac{1}{2}N & 0 & 0 & N & 0 & 0 \\ 0 & 54 & 13\ell_2 & 0 & 156 & -22\ell_2 \\ 0 & -13\ell_2 & -3\ell_2^2 & 0 & -22\ell_2 & 4\ell_2^2 \end{array}\right]\begin{Bmatrix} \ddot{u}_2 = 0 \\ \ddot{\bar{v}}_2 \\ \ddot{\theta}_2 \\ \ddot{\bar{u}}_3 = 0 \\ \ddot{\bar{v}}_3 = 0 \\ \theta_3 \end{Bmatrix}$$

Free vibr. eq.

$$\frac{1}{420}\begin{bmatrix} 156(m_1\ell_1 + m_2\ell_2) & 22(m_2\ell_2^2 - m_1\ell_1^2) & -13\, m_2\ell_2^2 \\ 22(m_2\ell_2^2 - m_1\ell_1^2) & 4(m_2\ell_2^3 + m_1\ell_1^3) & -3\, m_2\ell_2^3 \\ -13\, m_2\ell_2^3 & -3\, m_2\ell_2^3 & 4\, m_2\ell_2^3 \end{bmatrix}\begin{Bmatrix} \ddot{\bar{v}}_2 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix}$$

$$+ E\begin{bmatrix} 12\left(\frac{I_2}{\ell_2^3} + \frac{I_1}{\ell_1^3}\right) & 6\left(\frac{I_2}{\ell_2^2} - \frac{I_1}{\ell_1^2}\right) & 6\,\frac{I_2}{\ell_2^2} \\ 6\left(\frac{I_2}{\ell_2^2} - \frac{I_1}{\ell_1^2}\right) & 4\left(\frac{I_2}{\ell_2} + \frac{I_1}{\ell_1}\right) & 2\,\frac{I_2}{\ell_2} \\ 6\,\frac{I_2}{\ell_2^2} & 2\,\frac{I_2}{\ell_2} & 4\,\frac{I_2}{\ell_2} \end{bmatrix}\begin{Bmatrix} \bar{v}_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

10 – 17



From Eq. 10.2 – 1 and 10.2 – 10 the mass and stiffness matrices with associated displacement vector are:

$$M = \frac{m\ell}{420}\begin{bmatrix} 156 & 22\ell & 54 & -13\ell \\ 22\ell & 4\ell^2 & 13\ell & -3\ell^2 \\ 54 & 13\ell & 156 & -22\ell \\ -13\ell & -3\ell^2 & -22\ell & 4\ell^2 \end{bmatrix} \qquad \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

$$K = \frac{EI}{\ell^3}\begin{bmatrix} 12 & 6\ell & -12 & 6\ell \\ 6\ell & 4\ell^2 & -6\ell & 2\ell^2 \\ -12 & -6\ell & 12 & -6\ell \\ 6\ell & 2\ell^2 & -6\ell & 4\ell^2 \end{bmatrix}$$

For the pinned-free beam $v_1 = 0$ which eliminates the first column, and for the eigen problem, F_{1x} is of no interest so that the first row can also be deleted. Also when the charistic equation $|-\omega^2 M + K| = 0$ is solved, the ℓ in the matrices factor out, and we can let $\ell = 1.0$ and rewrite the equation as:

$$\left| -\left(\frac{\omega^2 m\ell}{420}\right)\begin{bmatrix} 4 & 13 & -3 \\ 13 & 156 & -22 \\ -3 & -22 & 4 \end{bmatrix} + \left(\frac{EI}{\ell^3}\right)\begin{bmatrix} 4 & -6 & 2 \\ -6 & 12 & -6 \\ 2 & -6 & 4 \end{bmatrix} \right| = 0$$

It should be noted that the stiffness matrix has a determinant which is zero, and therefore cannot be inverted or decomposed. It is therefore necessary to decompose the M matrix.

10-17 cont

```
»m=[ 4 13 -3 ; 13 156 -22; -3 -22 4]

m =

     4    13    -3
    13   156   -22
    -3   -22     4

»k=[ 4 -6 2; -6 12 -6; 2 -6 4]

k =

     4    -6     2
    -6    12    -6
     2    -6     4

»q=chol(m)

q =

    2.0000    6.5000   -1.5000
         0   10.6654   -1.1486
         0         0    0.6563

»qin=inv(q)

qin =

    0.5000   -0.3047    0.6094
         0    0.0938    0.1641
         0         0    1.5236

»a=qin'*k*qin

a =

    1.0000   -0.8907    2.2503
   -0.8907    0.8198   -2.3868
    2.2503   -2.3868   10.6088
```

10-17 cont.

```
»[v,d]=eig(a)

v =

    0.5916    0.7760    0.2186
    0.8043   -0.5495   -0.2262
    0.0555   -0.3096    0.9492


d =
    λ1        λ2        λ3
    0.0000         0         0
         0    0.7329         0
         0         0   11.6957

»x=qin*v

x =

    0.0845    0.3668    0.7567
    0.0845   -0.1023    0.1345
    0.0845   -0.4717    1.4463

»
```

$$\lambda = \left(\frac{\omega^2 m \ell^4}{420EI}\right)$$

$\omega_1 = 0$

$\omega_2 = 17.54\sqrt{\frac{EI}{m\ell^4}}$

$\omega_3 = 70.39\sqrt{\frac{EI}{m\ell^4}}$

exact values

$\omega_0 = 0$

$\omega_1 = 15.4\sqrt{\frac{EI}{m\ell^4}}$

$\omega_2 = 49.96\sqrt{\frac{EI}{m\ell^4}}$

Conclusion: One element model is unsatisfactory

Alternative: use the program beam.m see the next problem for details about the use of this code.

10-18

```
»
 What is the length of the bar?  1

len =

     1


 How many elements would you like? 2

n =

     2


 What is the mass of the beam ? 1

mass =

     1


 What is the elastic modulus of the beam ? 1

E =

     1


 What is the moment of inertia of the beam?1

I =

     1


v =

    0.1311    0.4801   -0.6584    0.5245    0.2092
   -0.0266   -0.3339    0.3855    0.5244    0.6813
    0.3884    0.6812    0.4446   -0.2905    0.3211
   -0.2459   -0.1322   -0.4223   -0.5961    0.6233
    0.8779   -0.4202   -0.2049   -0.1008    0.0219
```

10-18 cont

lambda =

1.0e+04 *

6.2116	0	0	0	0
0	1.8376	0	0	0
0	0	0.3116	0	0
0	0	0	0.0241	0
0	0	0	0	0.0000

omega =

249.2306	0	0	0	0
0	135.5595	0	0	0
0	0	55.8228	0	0
0	0	0	15.5142	0
0	0	0	0	0.0000
		exact 49.96	15.4	0

x =

22.2515	29.2961	-13.3294	5.4582	1.7321
0.4353	-1.1285	0.4848	1.1871	0.8660
28.3733	10.7305	11.8711	-2.7048	1.7321
3.3798	-2.2721	-2.0447	-2.0225	1.7321
69.3715	-33.2071	-16.1938	-7.9688	1.7321

10-19

Input $\ell = 1$
$n = 6$
$mass = 1$
$E = 1$
$I = 1$

10-19 cont

omega =

1.0e+03 *

Columns 1 through 4

2.1561	0	0	0
0	1.4025	0	0
0	0	1.7392	0
0	0	0	1.0621
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

Columns 5 through 8

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0.7901	0	0	0
0	0.5804	0	0
0	0	0.4248	0
0	0	0	0.2782
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

Columns 9 through 12

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

10-19 cont

0	0	0	0
0	0	0	0
0	0	0	0
0.1805	0	0	0
0	0.1048	0	0
0	0	0.0500	0
0	0	0	0.0154
0	0	0	0
		exact 49.96	exact 15.4

Column 13

0
0
0
0
0
0
0
0
0
0
0
0
0 + 0.0000i

exact 0

x =

Columns 1 through 7

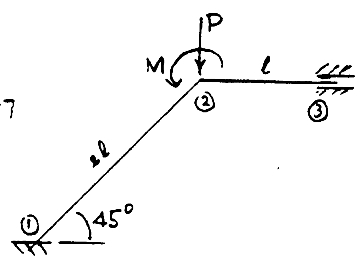
10.1121	-99.2702	-129.9937	-75.3367	57.9828	45.6155	37.4875
0.0505	0.8057	0.1529	1.2468	-1.2426	-0.8243	-0.1698
12.3264	-60.8796	-123.0672	-9.3374	-22.3957	-36.0458	-36.9790
0.1232	0.9882	0.2894	0.3091	0.9599	1.3027	0.3350
19.9391	24.5986	-103.0258	73.0222	-40.6822	11.3519	35.4673
0.2497	0.4064	0.3951	-1.1702	0.5011	-1.2346	-0.4910
36.2843	91.0508	-72.0052	27.4385	53.8227	18.1045	-32.9921
0.4865	-0.4901	0.4590	-0.5987	-1.3465	0.6475	0.6353
68.5425	87.0711	-33.3062	-66.2113	-0.8834	-39.9837	29.6521
0.9946	-1.0285	0.4871	1.0447	0.5653	0.1777	-0.7220
132.3452	15.2116	9.2843	-43.2898	-52.5323	44.3715	-24.7305
5.9261	-2.1565	1.3595	2.2411	2.2287	-2.1771	1.8257
358.7717	-103.5514	74.0764	89.3228	71.9851	-56.8220	39.1004

Columns 8 through 13

10-19 cont

-23.3039	-19.2839	-14.5747	10.0327	5.4017	1.7321
-0.5933	-1.1557	-1.4162	1.3123	0.8337	0.2887
21.4342	11.7057	1.8988	3.8566	4.2193	1.7321
1.0915	1.4029	0.3698	1.0115	1.2992	0.5774
-16.1249	5.0703	14.0933	-7.0208	1.1312	1.7321
-1.4142	-0.5490	1.3266	-0.5131	1.1698	0.8660
8.2338	-17.8814	-5.4988	-9.0513	-2.6879	1.7321
1.5151	-0.7502	-0.6786	-1.3227	0.4390	1.1547
1.0618	16.4539	-12.2908	0.6700	-5.8784	1.7321
-1.2954	1.3320	-0.9506	-0.2342	-0.7036	1.4434
-8.9312	-3.7942	10.7315	11.4386	-7.5543	1.7321
2.0371	-2.0418	2.0180	2.0047	-2.0005	1.7321
34.3242	-27.4611	20.6364	14.1729	-7.8613	1.7321

10-20



Element 1-2 $\quad \alpha = 45^\circ \quad C = .707 \quad S = .707$

$$c^2 = s^2 = cs = .50$$

$$\ell_{12} = 2\ell$$

$$\bar{k} = \left(\frac{EI}{8\ell^3}\right)\begin{bmatrix} (.5R_{1-2}+6) & .50(R_{1-2}-12) & 12\ell\cdot .707 \\ .50(R_{1-2}-12) & .5(R_{1-2}+12) & -12\ell\cdot .707 \\ 12\ell\cdot .707 & -12\ell\cdot .707 & 16\ell^2 \end{bmatrix}\begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ \theta_2 \end{Bmatrix} \qquad \text{All } R_{1-2} = \frac{A}{I}(2\ell)^2 = \frac{4A\ell^2}{I}$$

Element 2-3 $\quad \alpha = 0 \quad C = 1, \; S = 0, \; \ell_{23} = \ell$

$$\bar{k} = \left(\frac{EI}{\ell^3}\right)\begin{bmatrix} R_{2-3} & 0 & 0 \\ 0 & 12 & 6\ell \\ 0 & 6\ell & 4\ell^2 \end{bmatrix}\begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ \theta_2 \end{Bmatrix} \qquad \text{All } R_{2-3} = \frac{A\ell^2}{I} = \frac{1}{4}R_{1-2}$$

Assembled with all $R_s = R_{12} = R$

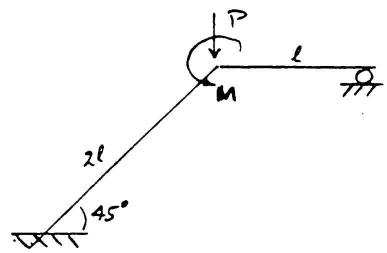
$$\bar{k} = \left(\frac{EI}{8\ell^3}\right)\begin{bmatrix} (.5R+2R+6) & .5(R-12) & .707\times 12\ell \\ .5(R+12) & .5(R+12)+96 & -.707\times 12\ell + 48\ell \\ 12\times .707\,\ell & -.707\times 12\,\ell + 48\ell & 16\ell^2 + 32\ell^2 \end{bmatrix}\begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ \theta_2 \end{Bmatrix}$$

10-20 CONT

$$\begin{Bmatrix} 0 \\ -P \\ M \end{Bmatrix} = \left(\frac{EI}{8\ell^3}\right)\begin{bmatrix} (2.5R+6) & .5(R-12) & 8.484\ell \\ .5(R-12) & (.5R+102) & 39.52\ell \\ 8.484\ell & 39.52\ell & 48\ell^2 \end{bmatrix}\begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ \theta_2 \end{Bmatrix}$$

$$R = \frac{4A\ell^2}{I}$$

10-21



Element 1-2

$$\bar{k} = \left(\frac{EI}{8\ell^3}\right)\begin{bmatrix} (.5R+6) & .5(R-12) & 8.484\ell \\ .5(R-12) & .5(R+12) & -8.484\ell \\ 8.484\ell & -8.484\ell & 16\ell^2 \end{bmatrix}$$

Element 2-3

$$\bar{k} = \left(\frac{EI}{\ell^3}\right)\begin{bmatrix} R_{2-3} & 0 & 0 \\ 0 & 3 & 3\ell \\ 0 & 3\ell & 3\ell^2 \end{bmatrix} = \left(\frac{EI}{8\ell^3}\right)\begin{bmatrix} 2R & 0 & 0 \\ 0 & 24 & 24\ell \\ 0 & 24\ell & 24\ell^2 \end{bmatrix}$$

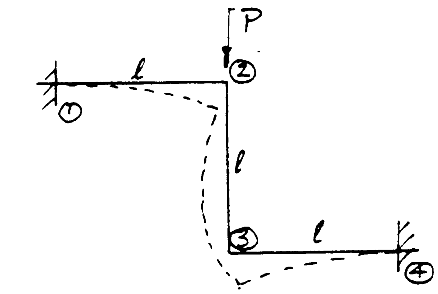
Assembled

$$\begin{Bmatrix} 0 \\ -P \\ M \end{Bmatrix} = \left(\frac{EI}{8\ell^3}\right)\begin{bmatrix} (2.5R+6) & .5(R-12) & 8.484\ell \\ .5(R-12) & (.5R+30) & 15.52\ell \\ 8.484\ell & 15.52\ell & 40\ell^2 \end{bmatrix}\begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ \theta_2 \end{Bmatrix}$$

From 1st of above equation express $\bar{u}_2$ in terms of $\bar{v}_2$ and θ_2 and write

$$\begin{Bmatrix} -P \\ M \end{Bmatrix} = \left(\frac{EI}{8\ell^3}\right)\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\begin{Bmatrix} \bar{v}_2 \\ \theta_2 \end{Bmatrix}$$

10-22 a)



$$\begin{Bmatrix} \bar{F}_{1x} \\ \vdots \\ \bar{M}_4 \end{Bmatrix} = \frac{EI}{\ell^3}\begin{bmatrix} 12 \times 12 \end{bmatrix}\begin{Bmatrix} \bar{u}_1 \\ \vdots \\ \theta_4 \end{Bmatrix}$$

Elements 1-2 and 3-4 $\alpha = 0$ $C = 1$, $S = 0$

$$\bar{k}_{1-2} = \bar{k}_{3-4} = \frac{EI}{\ell^3}\left[\begin{array}{ccc|ccc} R & 0 & 0 & -R & 0 & 0 \\ 0 & 12 & 6\ell & 0 & -12 & 6\ell \\ 0 & 6\ell & 4\ell^2 & 0 & -6\ell & 2\ell^2 \\ \hline -R & 0 & 0 & R & 0 & 0 \\ 0 & -12 & -6\ell & 0 & 12 & -6\ell \\ 0 & 6\ell & 2\ell^2 & 0 & -6\ell & 4\ell^2 \end{array}\right] \quad \text{vectors} \left\{\begin{array}{c} \bar{u}_1 = 0 \\ \bar{v}_1 = 0 \\ \theta_1 = 0 \\ \hline \bar{u}_2 \\ \bar{v}_2 \\ \theta_2 \end{array}\right\} \text{or} \left\{\begin{array}{c} \bar{u}_3 \\ \bar{v}_3 \\ \theta_3 \\ \hline \bar{u}_4 = 0 \\ \bar{v}_4 = 0 \\ \theta_4 = 0 \end{array}\right\}$$

Element 2-3 $\alpha = 270°$ or $-90°$ $C = 0$, $S = -1$

$$\bar{k}_{2-3} = \frac{EI}{\ell^3}\left[\begin{array}{ccc|ccc} 12 & 0 & 6\ell & -12 & 0 & 6\ell \\ 0 & R & 0 & 0 & -R & 0 \\ 6\ell & 0 & 4\ell^2 & -6\ell & 0 & 2\ell^2 \\ \hline -12 & 0 & -6\ell & 12 & 0 & -6\ell \\ 0 & -R & 0 & 0 & R & 0 \\ 6\ell & 0 & 2\ell^2 & -6\ell & 0 & 4\ell^2 \end{array}\right] \quad \text{vector} \left\{\begin{array}{c} \bar{u}_2 \\ \bar{v}_2 \\ \theta_2 \\ \hline \bar{u}_3 \\ \bar{v}_3 \\ \theta_3 \end{array}\right\}$$

Only displacements involving ② and ③ need to be considered. Thus we need only to add lower right quarter of $\bar{k}_{1-2}$ to upper left quarter of k_{2-3}, and upper left qtr. of k_{3-4} to lower right qtr. of $\bar{k}_{2-3}$

$$\left\{\begin{array}{c} \bar{F}_{2x} = 0 \\ -P \\ \bar{M}_2 = 0 \\ \hline \bar{F}_{3x} = 0 \\ \bar{F}_{3y} = 0 \\ \bar{M}_3 = 0 \end{array}\right\} = \frac{EI}{\ell^3}\left[\begin{array}{ccc|ccc} \cancel{(R+12)} & \cancel{0} & \cancel{6\ell} & \cancel{-12} & \cancel{0} & \cancel{6\ell} \\ 0 & (R+12) & -6\ell & 0 & -R & 0 \\ 6\ell & -6\ell & 8\ell^2 & -6\ell & 0 & 2\ell^2 \\ \hline \cancel{-12} & \cancel{0} & \cancel{-6\ell} & \cancel{(R+12)} & \cancel{0} & \cancel{-6\ell} \\ 0 & -R & 0 & 0 & (R+12) & 6\ell \\ 6\ell & 0 & 2\ell^2 & -6\ell & 6\ell & 8\ell^2 \end{array}\right]\left\{\begin{array}{c} \bar{u}_2 \\ \bar{v}_2 \\ \theta_2 \\ \hline \bar{u}_3 \\ \bar{v}_3 \\ \theta_3 \end{array}\right\}$$

symmetric matrix

10-22(a) Cont.

For no axial extension $\bar{u}_2 = -\bar{u}_3 = 0$

this eliminates 1st and 4th columns

Then $F_{2x} = F_{3x} = 0$ and the 1st and 4th rows

requires that $\theta_3 = -\theta_2$ which eliminates

the 1st and 4th rows, leaving the 4×4

matrix

$$\begin{Bmatrix} -P \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \frac{EI}{\ell^3}\begin{bmatrix} (R+12) & -6\ell & -R & 0 \\ -6\ell & 8\ell^2 & 0 & 2\ell^2 \\ -R & 0 & (R+12) & 6\ell \\ 0 & 2\ell^2 & 6\ell & 8\ell^2 \end{bmatrix}\begin{Bmatrix} \bar{v}_2 \\ \theta_2 \\ \bar{v}_3 \\ \theta_3 \end{Bmatrix}$$

symmetric

Since $\bar{v}_3 = \bar{v}_2$ add 3rd column to 1st column.

Also since $\theta_3 = -\theta_2$, subtract 4th col. from 2nd col.

This also reduces the matrix to a 2×2

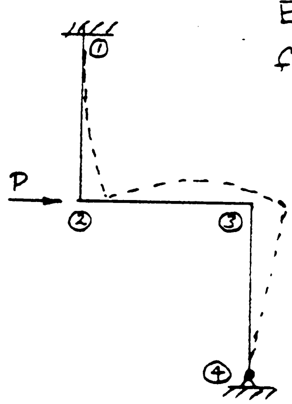
$$\begin{Bmatrix} -P \\ 0 \end{Bmatrix} = \frac{EI}{\ell^3}\begin{bmatrix} 12 & -6\ell \\ -6\ell & 6\ell^2 \end{bmatrix}\begin{Bmatrix} \bar{v}_2 \\ \theta_2 \end{Bmatrix}$$

Solving the two equations

$$\ell\theta_2 = \bar{v}_2$$

$$\boxed{\bar{v}_2 = \frac{-P\ell^3}{6EI} \qquad \theta_2 = -\frac{P\ell^2}{6EI}}$$

10-22(b)



Elements 1-2 and 2-3 are already available from Prob 10-2-(a) by interchanging 1-2 and 2-3

The new element 3-4 is obtained.

with $\alpha = -90^\circ$ $C=0$, $S=-1$

Element 3-4

$$\frac{EI}{l^3}\left[\begin{array}{ccc|ccc} 3 & 0 & 3l & -3 & 0 & 3l \\ 0 & R & 0 & 0 & -R & 0 \\ 3l & 0 & 3l^2 & 0 & 0 & 0 \\ \hline -3 & 0 & -3l & 3 & 0 & -3l \\ 0 & -R & 0 & 0 & R & 0 \\ 0 & 0 & 0 & -3l & 0 & 3l^2 \end{array}\right]$$

$$\left\{\begin{array}{c} \bar F_{1x} \\ \bar F_{1y} \\ \bar M_1 \\ \hline \bar F_{2x}=P \\ \bar F_{2y}=0 \\ \bar M_2=0 \\ \hline \bar F_{3x}=0 \\ \bar F_{3y}=0 \\ \bar M_3=0 \\ \hline \bar F_{4x} \\ \bar F_{4y} \\ \bar M_4 \end{array}\right\} = \frac{EI}{l^3}\left[\begin{array}{ccc|ccc|ccc|ccc} & & & -12 & 0 & 6l & & & & & & \\ & & & 0 & -R & 0 & & & & & & \\ & & & -6l & 0 & 2l^2 & & & & & & \\ \hline & & & (12+R) & 0 & -6l & -R & 0 & 0 & & & \\ & & & 0 & (12+R) & 6l & 0 & -12 & 6l & & & \\ & & & -6l & 6l & 8l^2 & 0 & -6l & 2l^2 & & & \\ \hline & & & -R & 0 & 0 & (R+3) & 0 & 3l & -3 & 0 & 3l \\ & & & 0 & -12 & -6l & 0 & (R+12) & -6l & 0 & -R & 0 \\ & & & 0 & 6l & 2l^2 & 3l & -6l & 7l^2 & 0 & 0 & 0 \\ \hline & & & & & & -3 & 0 & -3l & 3 & 0 & -3l \\ & & & & & & 0 & -R & 0 & 0 & R & 0 \\ & & & & & & 0 & 0 & 0 & -3l & 0 & 3l^2 \end{array}\right]\left\{\begin{array}{c} \\ 0 \\ \\ \hline \bar u_2 \\ \bar v_2 \\ \theta_2 \\ \hline \bar u_3 \\ \bar v_3 \\ \theta_3 \\ \hline \\ 0 \\ \\ \end{array}\right\}$$

We need only the central part of the matrix

10 – 22(b) Cont. $\bar{v}_2 = \bar{v}_3 = 0$ ∴ eliminate cols 2 & 5

$$\begin{Bmatrix} \bar{F}_{2x}=P \\ \bar{M}_2 \\ \bar{F}_{3x} \\ \bar{M}_3 \end{Bmatrix} = \frac{EI}{\ell^3}\begin{bmatrix} (R+12) & -6\ell & -R & 0 \\ -6\ell & 8\ell^2 & 0 & 2\ell^2 \\ -R & 0 & (R+3) & 3\ell \\ 0 & 2\ell^2 & 3\ell & 7\ell^2 \end{bmatrix}\begin{Bmatrix} \bar{u}_2 \\ \theta_2 \\ \bar{u}_3 \\ \theta_3 \end{Bmatrix}$$

$\bar{u}_2 = \bar{u}_3$ ∴ add col 3 to 1

$$\begin{Bmatrix} \frac{P\ell^3}{EI} \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 12 & -6\ell & 0 \\ -6\ell & 8\ell^2 & 2\ell^2 \\ 3 & 0 & 3\ell \\ 3\ell & 2\ell^2 & 7\ell^2 \end{bmatrix}\begin{Bmatrix} u_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix}$$

Eq. no.

$\frac{P\ell^3}{EI} = 12\bar{u}_2 - 6\ell\theta_2$ (1)

$-6\ell\bar{u}_2 + 8\ell^2\theta_2 + 2\ell^2\theta_3 = 0$ (2)

$3\bar{u}_2 + 3\ell\theta_3 = 0$ (3)

$3\ell\bar{u}_2 + 2\ell^2\theta_2 + 7\ell^2\theta_3 = 0$ (4)

$2\times(4) + (2) = \; 0u_2 + 12\ell^2\theta_2 + 16\ell^2\theta_3 = 0$ ∴ $\theta_2 = -\frac{4}{3}\theta_3$

(3) gives $\bar{u}_2 = -\ell\theta_3$ ∴ $= \frac{3}{4}\ell\theta_2$

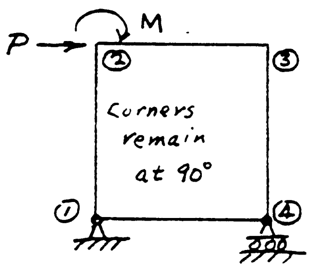
Subst $\theta_2 = -\frac{4}{3}\theta_3$ into (1)

$$\frac{P\ell^3}{EI} = 12u_2 - 6\ell\left(-\frac{4}{3}\theta_3\right) = 12\bar{u}_2 + 8\ell\theta_3 = 12\bar{u}_2 - 8\bar{u}_2 = 4\bar{u}_2$$

$$\therefore \quad \bar{u}_2 = \frac{P\ell^3}{4EI}, \qquad \theta_3 = -\frac{P\ell^2}{4EI}, \qquad \theta_2 = \frac{P\ell^2}{3EI}$$

Comparing with Prob (10–22(a)) the deflection under P is larger with pinned end, and the two rotations are dissimular and also larger. Conclusion: Pinned end leads to more flexible structure.

10-22 (c)



Element

1-2	$c = 0,\ s = 1$
2-3	$c = 1,\ s = 0$
3-4	$c = 0,\ s = -1$
4-1	$c = -1\ \ s = 0$

Element stiffness

$$
1\text{-}2 \left[\begin{array}{ccc|ccc}
12 & 0 & -6l & -12 & 0 & -6l \\
0 & R & 0 & 0 & -R & 0 \\
-6l & 0 & 4l^2 & 6l & 0 & 2l^2 \\
\hline
-12 & 0 & 6l & 12 & 0 & 6l \\
0 & -R & 0 & 0 & R & 0 \\
-6l & 0 & 2l^2 & 6l & 0 & 4l^2
\end{array}\right]
\qquad
2\text{-}3 \left[\begin{array}{ccc|ccc}
R & 0 & 0 & -R & 0 & 0 \\
0 & 12 & 6l & 0 & -12 & 6l \\
0 & 6l & 4l^2 & 0 & -6l & 2l^2 \\
\hline
-R & 0 & 0 & R & 0 & 0 \\
0 & -12 & -6l & 0 & 12 & -6l \\
0 & 6l & 2l^2 & 0 & -6l & 4l^2
\end{array}\right] = 1\text{-}4
$$

$$
3\text{-}4 \left[\begin{array}{ccc|ccc}
12 & 0 & 6l & -12 & 0 & 6l \\
0 & R & 0 & 0 & -R & 0 \\
6l & 0 & 4l^2 & -6l & 0 & 2l^2 \\
\hline
-12 & 0 & -6l & 12 & 0 & -6l \\
0 & -R & 0 & 0 & R & 0 \\
6l & 0 & 2l^2 & -6l & 0 & 4l^2
\end{array}\right]
\qquad
4\text{-}1 \left[\begin{array}{ccc|ccc}
R & 0 & 0 & -R & 0 & 0 \\
0 & 12 & -6l & 0 & -12 & -6l \\
0 & -6l & 4l^2 & 0 & 6l & 2l^2 \\
\hline
-R & 0 & 0 & R & 0 & 0 \\
0 & -12 & 6l & 0 & 12 & 6l \\
0 & -6l & 2l^2 & 0 & 6l & 4l^2
\end{array}\right]
$$

Displacements.

$\bar{u}_1 = \bar{v}_1 = 0$

$\bar{v}_2 = \bar{v}_3 = 0$

$\bar{u}_2 = \bar{u}_3$

$\bar{v}_4 = 0$

$\bar{u}_4 \cong 0$

Coordinates not zero

$\bar{u}_2,\ \bar{u}_3,\ \theta_1,\ \theta_2,\ \theta_3,\ \theta_4$

none of the θ_s need to be equal, however we can assume that $\bar{u}_2 = \bar{u}_3$

This leaves 5 unknowns.

10-22 (c) Cont.

Assembled matrix (symmetric)

$$
\begin{array}{c}
① \\ \\ \\ ② \\ \\ \\ ③ \\ \\ \\ ④ \\ \\ \\
\end{array}
\begin{bmatrix}
(12+R) & 0 & -6l & -12 & 0 & -6l & & & & -R & 0 & 0 \\
0 & (12+R) & 6l & 0 & -R & 0 & & & & 0 & -12 & 6l \\
-6l & 6l & 8l^2 & 6l & 0 & 2l^2 & & & & 0 & -6l & 2l^2 \\
-12 & 0 & 6l & (12+R) & 0 & 6l & -R & 0 & 0 & & & \\
0 & -R & 0 & 0 & (12+R) & 6l & 0 & -12 & 6l & & & \\
-6l & 0 & 2l^2 & 6l & 6l & 8l^2 & 0 & -6l & 2l^2 & & & \\
 & & & -R & 0 & 0 & (12+R) & 0 & 6l & -12 & 0 & 6l \\
 & & & 0 & -12 & -6l & 0 & (12+R) & -6l & 0 & -R & 0 \\
 & & & 0 & 6l & 2l^2 & 6l & -6l & 8l^2 & -6l & 0 & 2l^2 \\
-R & 0 & 0 & & & & -12 & 0 & -6l & (12+R) & 0 & -6l \\
0 & -12 & -6l & & & & 0 & -R & 0 & 0 & (12+R) & -6l \\
0 & 6l & 2l^2 & & & & 6l & 0 & 2l^2 & -6l & -6l & 8l^2
\end{bmatrix}
\begin{Bmatrix}
0 \\ 0 \\ \theta_1 \\ u_2 \\ 0 \\ \theta_2 \\ \bar{u}_3 = \bar{u}_2 \\ 0 \\ \theta_3 \\ 0 \\ 0 \\ \theta_4
\end{Bmatrix}
\quad
\begin{Bmatrix}
\bar{u}_1 = 0 \\ \bar{v}_1 = 0 \\ \theta_1 \\ \bar{u}_2 \\ \bar{v}_2 = 0 \\ \theta_2 \\ \bar{u}_3 \\ \bar{v}_3 = 0 \\ \theta_3 \\ \bar{u}_4 = 0 \\ \bar{v}_4 = 0 \\ \theta_4
\end{Bmatrix}
$$

$\bar{u}_3 = \bar{u}_2$ $\therefore$ add col. 7 to 4 and rewrite

$$
\begin{Bmatrix}
\bar{F}_{1x} \\ \bar{F}_{1y} \\ 0 \\ \bar{P} \\ \bar{0} \\ -M \\ \bar{0} \\ \bar{0} \\ 0 \\ \bar{F}_{4x} \\ \bar{F}_{4y} \\ 0
\end{Bmatrix}
= \frac{EI}{l^3}
\begin{bmatrix}
-6l & -12 & -6l & 0 & 0 \\
6l & 0 & 0 & 0 & 6l \\
8l^2 & 6l & 2l^2 & 0 & 2l^2 \\
6l & 12 & 6l & 0 & 0 \\
0 & 0 & 6l & 6l & 0 \\
2l^2 & 6l & 8l^2 & 2l^2 & 0 \\
0 & 12 & 0 & 6l & 6l \\
0 & 0 & -6l & -6l & 0 \\
0 & 6l & 2l^2 & 8l^2 & 2l^2 \\
0 & -12 & 0 & -6l & -6l \\
-6l & 0 & 0 & 0 & -6l \\
2l^2 & 6l & 0 & 2l^2 & 8l^2
\end{bmatrix}
\begin{Bmatrix}
\theta_1 \\ \bar{u}_2 \\ \theta_2 \\ \theta_3 \\ \theta_4
\end{Bmatrix}
$$

12x 5 matrix

5th and 8th equations are identical — both give $\theta_2 = -\theta_3$

(remove 5th & 8th eqs. after subtracting 4th col from 3rd col.)

10-22 (c) Cont Rewrite eq with col 4 subtracted from col 3 and 5^{th} & 8^{th} eqs. removed

$$\begin{Bmatrix} \bar{F}_{1x} \\ \bar{F}_{1y} \\ 0 \\ P \\ -M \\ 0 \\ 0 \\ \bar{F}_{4x} \\ \bar{F}_{4y} \\ 0 \end{Bmatrix} = \frac{EI}{\ell^3} \begin{bmatrix} -6\ell & -12 & -6\ell & 0 \\ 6\ell & 0 & 0 & 6\ell \\ 8\ell^2 & 6\ell & 2\ell^2 & 2\ell^2 \\ 6\ell & 12 & 6\ell & 0 \\ 2\ell^2 & 6\ell & 6\ell^2 & 0 \\ 0 & 12 & -6\ell & 6\ell \\ 0 & 6\ell & -6\ell^2 & 2\ell^2 \\ 0 & -12 & 6\ell & -6\ell \\ -6\ell & 0 & 0 & -6\ell \\ 2\ell^2 & 6\ell & -2\ell^2 & 8\ell^2 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \bar{u}_2 \\ \theta_2 \\ \theta_4 \end{Bmatrix}$$

10 x 4 matrix

Eqs. for $\bar{F}_{1y}$ and $\bar{F}_{4y}$ are equal but of opposite signs

$$\therefore \quad \bar{F}_{1y} = -\bar{F}_{4y} \qquad \text{(remove these two eqs.)}$$

6^{th} eq and $\bar{F}_{4x}$ are of opposite signs but 6^{th} eq $= 0$

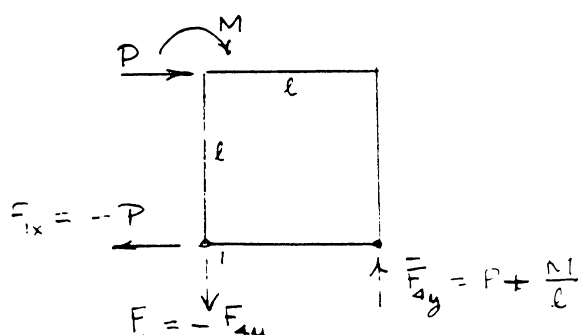
$$\therefore \quad \bar{F}_{4x} = 0 \qquad \text{(remove these two eqs)}$$

Eqs for $\bar{F}_{1x}$ and P are of opposite signs

$$\therefore \quad F_{1x} = -P$$

Moment about ① gives

$$F_{4y}\,\ell = P\ell + M$$



The above is just for checking, since these results can be obtained from statics.

10-22 (c) Cont.

Rewrite unused eqs from 10x4 matrix

$$\frac{P\ell^3}{EI} = 6\ell\theta_1 + 12\bar{u}_2 + 6\ell\theta_2 \qquad (1)$$

$$-\frac{M\ell^2}{EI} = 2\ell\theta_1 + 6\bar{u}_2 + 6\ell\theta_2 \qquad (2)$$

$$0 = 8\ell^2\theta_1 + 6\ell\bar{u}_2 + 2\ell^2\theta_2 + 2\ell^2\theta_4 \qquad (3)$$

$$0 = 2\ell^2\theta_1 + 6\ell\bar{u}_2 - 2\ell^2\theta_2 + 8\ell^2\theta_4 \qquad (4)$$

Eliminate θ_4 betw. (3) & (4)

$$0 = 30\ell^2\theta_1 + 18\ell\bar{u}_2 + 10\ell^2\theta_2 \qquad (5)$$

Eqs. (1) (2) & (5) has 3 unknowns with 3 eqs.

In (5) solve for $\ell\theta_1$ in terms of $\bar{u}_2$ and θ_2

$$\ell\theta_1 = -(.60\bar{u}_2 + .3333\ell\theta_2) \quad \text{subst. into (1) and (2)}$$

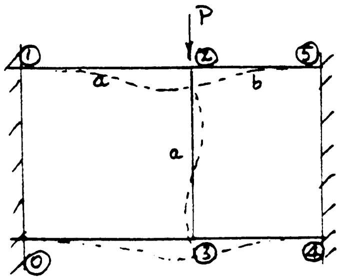
$$\frac{P\ell^3}{EI} = (12 - 3.60)\bar{u}_2 + (6\ell - 2\ell)\theta_2 = 8.40\bar{u}_2 + 4\ell\theta_2$$

$$-\frac{M\ell^2}{EI} = (6 - 1.20)\bar{u}_2 + (6\ell - .666\ell)\theta_2 = 4.80\bar{u}_2 + 5.333\ell\theta_2$$

Eliminate θ_2 for eq. for $\bar{u}_2$, then eliminate $\bar{u}_2$

$$\boxed{\begin{aligned} \bar{u}_2 &= \frac{1}{4.80}\left(\frac{P\ell^3}{EI} + .75\frac{M\ell^2}{EI}\right) \\ \theta_2 &= \frac{-1}{5.333\ell}\left(\frac{P\ell^3}{EI} + 1.75\frac{M\ell^2}{EI}\right) \end{aligned}}$$

10-2) (d)



Let $a = \ell_1 = \ell$

$b = \ell_2 = \frac{b}{a}\ell_1$

$\bar{u}_2 = \bar{u}_3 = 0$

$\bar{v}_2 = \bar{v}_3$

$$\left\{\begin{array}{l} \bar{u}_1 = 0 \\ \bar{v}_1 = 0 \\ \theta_1 = 0 \\ \hline \bar{u}_2 \\ \bar{v}_2 \\ \theta_2 \\ \hline \bar{u}_3 \\ \bar{v}_3 \\ \theta_3 \\ \hline \bar{u}_4 = 0 \\ \bar{v}_4 = 0 \\ \theta_4 = 0 \end{array}\right\}$$

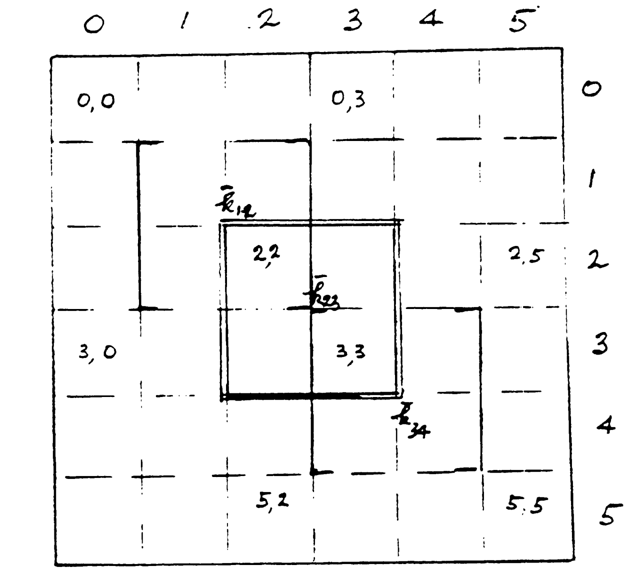
$$\bar{k}_{1-2} = \frac{EI}{\ell_1^3}\left[\begin{array}{ccc|ccc} & & & -R & 0 & 0 \\ & & & 0 & -12 & 6\ell \\ - & - & & 0 & -6\ell & 2\ell^2 \\ \hline & & & R & 0 & 0 \\ & & & 0 & 12 & -6\ell \\ & & & 0 & -6\ell & 4\ell^2 \end{array}\right] = \bar{k}_{0-3}$$

$$\bar{k}_{2-3} = \frac{EI}{\ell_1^3}\left[\begin{array}{ccc|ccc} 12 & 0 & 6\ell & -12 & 0 & 6\ell \\ 0 & R & 0 & 0 & -R & 0 \\ 6\ell & 0 & 4\ell^2 & -6\ell & 0 & 2\ell^2 \\ \hline -12 & 0 & -6\ell & 12 & 0 & -6\ell \\ 0 & -R & 0 & 0 & R & 0 \\ 6\ell & 0 & 2\ell^2 & -6\ell & 0 & 4\ell^2 \end{array}\right]$$

$$\bar{k}_{3-4} = \frac{EI}{\ell_2^3}\left[\begin{array}{ccc|ccc} R & 0 & 0 & & & \\ 0 & 12 & 6\ell_2 & & & \\ 0 & 6\ell_2 & 4\ell_2^2 & - & - & - \\ \hline -R & 0 & 0 & & & \\ 0 & -12 & -6\ell_2 & & & \\ 0 & 6\ell_2 & 2\ell_2^2 & & & \end{array}\right] = \frac{EI}{\ell^3}\left[\begin{array}{ccc|ccc} R(\frac{a}{b})^3 & 0 & 0 & & & \\ 0 & 12(\frac{a}{b})^3 & 6\ell(\frac{a}{b})^2 & & & \\ 0 & 6\ell(\frac{a}{b})^2 & 4\ell^2(\frac{a}{b}) & - & - & - \\ \hline -R(\frac{a}{b})^3 & 0 & 0 & & & \\ 0 & -12(\frac{a}{b})^3 & -6\ell(\frac{a}{b})^2 & & & \\ 0 & 6\ell(\frac{a}{b})^2 & 2\ell^2(\frac{a}{b}) & & & \end{array}\right] = \bar{k}_{2-5} = \bar{k}_{3-4}$$

For the assembly we can first consider a matrix space of 6 x 6 each of which is composed of a 3x3 elements. i.e. $\bar{k}_{2-3}$ has 4 sub-matrices, each corner of which has a 3x3 matrix. Since ①, ⓪, ④ and ⑤ all have zero deflections, we will be concerned only with the displacements of ② and ③ which occupies the central portion of the 6x6 space.

10-22 (d) Cont.



sub matrices of $\bar{k}_{2-5}$

2,2 2,5

5,2 5,5

shown in 6 x 6 space

sub matrices of $\bar{k}_{0-3}$

0,0 0,3

3,0 3,3

shown in 6x6 space

The matrix of interest which is indicated by the square in double lines also involves the lower right quarter of $\bar{k}_{12}$, all of $\bar{k}_{23}$ and upper left quarter of $\bar{k}_{3-4}$

$$\frac{EI}{\ell^3}\left[\begin{array}{ccc|ccc}
\substack{R+12 \\ +R(\frac{a}{b})^3} & \substack{0+0 \\ 0} & \substack{0+6\ell \\ 0} & -12 & 0 & 6\ell \\
\substack{0+0 \\ 0} & \substack{12+R \\ 12(\frac{a}{b})^3} & \substack{-6\ell+0 \\ 6\ell(\frac{a}{b})^2} & 0 & -R & 0 \\
\substack{0+6\ell \\ 0} & \substack{-6\ell+0 \\ 6\ell(\frac{a}{b})^2} & \substack{4\ell^2+4\ell^2 \\ 4\ell^2(\frac{a}{b})} & -6\ell & 0 & 2\ell^2 \\
\hline
-12 & 0 & -6\ell & \substack{R+12 \\ R(\frac{a}{b})^3} & \substack{0+0 \\ 0} & \substack{0-6\ell \\ 0} \\
0 & -R & 0 & \substack{0+0 \\ 0} & \substack{12+R \\ 12(\frac{a}{b})^3} & \substack{-6\ell+0 \\ 6\ell(\frac{a}{b})^2} \\
6\ell & 0 & 2\ell^2 & 0-6\ell & \substack{-6\ell+0 \\ 6\ell(\frac{a}{b})^2} & \substack{4\ell^2+4\ell^2 \\ 4\ell^2(\frac{a}{b})}
\end{array}\right]
\begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ \theta_2 \\ \bar{u}_3 \\ \bar{v}_3 \\ \theta_3 \end{Bmatrix}$$

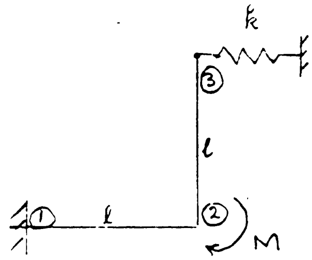
10-22 (d) Cont

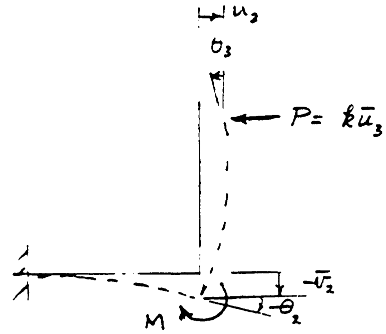
Note that $\begin{cases} \bar{u}_2 = \bar{u}_3 = 0 \\ \bar{v}_2 = \bar{v}_3 \\ \theta_2 = \theta_3 \end{cases}$ cross out Cols. 1 and 4; add col. 5 to 2; add col. 6 to 3

Resulting equations

$$\begin{Bmatrix} \bar{F}_{2y} = -P \\ \bar{M}_2 = 0 \end{Bmatrix} = \frac{EI}{a^3} \begin{bmatrix} (12)\left[1 + \left(\frac{a}{b}\right)^3\right] & 6\ell\left[\left(\frac{a}{b}\right)^2 - 1\right] \\ 6\ell\left[\left(\frac{a}{b}\right)^2 - 1\right] & 4\ell^2\left[2.5 + \left(\frac{a}{b}\right)\right] \end{bmatrix} \begin{Bmatrix} \bar{v}_2 \\ \theta_2 \end{Bmatrix}$$

10-22 (e)





$$\begin{Bmatrix} \bar{F}_{1x} \\ \bar{F}_{1y} \\ \bar{M}_1 \\ \bar{F}_{2x} \\ \bar{F}_{2y} \\ \bar{M}_2 \\ \bar{F}_{3x} \\ F_{3y} \\ \bar{M}_3 \end{Bmatrix} = \frac{EI}{\ell^3} \begin{bmatrix} R & 0 & 0 & -R & 0 & 0 & & & \\ 0 & 12 & 6\ell & 0 & -12 & 6\ell & & & \\ 0 & 6\ell & 4\ell^2 & 0 & -6\ell & 2\ell^2 & & & \\ -R & 0 & 0 & R+12 & 0+0 & 0-6\ell & -12 & 0 & -6\ell \\ 0 & -12 & -6\ell & 0+0 & 12+R & -6\ell+0 & 0 & -R & 0 \\ 0 & 6\ell & 2\ell^2 & 0-6\ell & -6\ell+0 & 4\ell^2+4\ell^2 & 6\ell & 0 & 2\ell^2 \\ & & & -12 & 0 & 6\ell & \left(12+\frac{k\ell^3}{EI}\right) & 0 & 6\ell \\ & & & 0 & -R & 0 & 0 & R & 0 \\ & & & -6\ell & 0 & 2\ell^2 & 6\ell & 0 & 4\ell^2 \end{bmatrix} \begin{Bmatrix} \bar{u}_1 = 0 \\ \bar{v}_1 = 0 \\ \theta_1 = 0 \\ \bar{u}_2 = 0 \\ \bar{v}_2 \\ \theta_2 \\ \bar{u}_3 \\ \bar{v}_3 \\ \theta_3 \end{Bmatrix}$$

$\bar{v}_3 = \bar{v}_2$ ∴ add col. 8 to col 5

$$\begin{Bmatrix} 0 \\ -M \\ 0 \\ 0 \end{Bmatrix} = \frac{EI}{\ell^3} \begin{bmatrix} 12 & -6\ell & 0 & 0 \\ -6\ell & 8\ell^2 & 6\ell & 2\ell^2 \\ 0 & 6\ell & \left(12+\frac{k\ell^3}{EI}\right) & 6\ell \\ 0 & 2\ell^2 & 6\ell & 4\ell^2 \end{bmatrix} \begin{Bmatrix} \bar{v}_2 \\ \theta_2 \\ \bar{u}_3 \\ \theta_3 \end{Bmatrix}$$ Sym. 4 x 4

4 eqs, 4 unknowns

10-22 (e) Cont.

$$① \quad 0 = 12\bar{v}_2 - 6\ell\theta_2$$

$$② \quad -\frac{M\ell^3}{EI} = -6\ell\bar{v}_2 + 8\ell^2\theta_2 + 6\ell\bar{u}_3 + 2\ell^2\theta_3$$

$$③ \quad 0 = 6\ell\theta_2 + \left(12+\frac{k\ell^3}{EI}\right)\bar{u}_3 + 6\ell\theta_3$$

$$④ \quad 0 = 2\ell^2\theta_2 + 6\ell\bar{u}_3 + 4\ell^2\theta_3$$

From ① $\quad \underline{\bar{v}_2 = \frac{\ell}{2}\theta_2}$ $\quad$ sub into ②

$$②' \quad -\frac{M\ell^3}{EI} = 5\ell^2\theta_2 + 6\ell\bar{u}_3 + 2\ell^2\theta_3$$

From ④ $\quad \underline{\theta_2 = -\frac{3}{\ell}\bar{u}_3 - 2\theta_3}$

Sub. into ③ $\quad 0 = -6\ell\left(\frac{3}{2}\bar{u}_3 + 2\theta_3\right) + \left(12+\frac{k\ell^3}{EI}\right)\bar{u}_3 + 6\ell\theta_3$

$$= \left(-18+12+\frac{k\ell^3}{EI}\right)\bar{u}_3 - 12\ell\theta_3 + 6\ell\theta_3$$

$$\therefore \quad \underline{6\ell\theta_3 = \left(-6+\frac{k\ell^3}{EI}\right)\bar{u}_3}$$

$$-\frac{M\ell^3}{EI} = 5\ell^2\left(-\frac{3}{\ell}\bar{u}_3 - 2\theta_3\right) + 6\ell\bar{u}_3 + 2\ell^2\theta_3 = -9\ell\bar{u}_3 - 8\ell^2\theta_3$$

Sub for θ_3 in above eq.

$$-\frac{M\ell^3}{EI} = -9\ell\bar{u}_3 - 8\ell^2\left(\frac{-6+\frac{k\ell^3}{EI}}{6\ell}\right)\bar{u}_3 = \left(-\ell - \frac{4}{3}\frac{k\ell^3}{EI}\right)\bar{u}_3$$

$$\therefore \quad \boxed{\bar{u}_3 = \frac{M\ell^2}{\left(1+\frac{4}{3}\frac{k\ell^3}{EI}\right)EI}} \qquad \boxed{\theta_3 = \frac{1}{6\ell}\frac{\left(-6+\frac{k\ell^3}{EI}\right)}{\left(1+\frac{4}{3}\frac{k\ell^3}{EI}\right)}\frac{M\ell^2}{EI}}$$

$$\boxed{\theta_2 = -\frac{3}{\ell}\bar{u}_3 - 2\theta_3} \qquad \boxed{\bar{v}_2 = -\frac{3}{2}\bar{u}_3 - \ell\theta_2}$$

10–22 (f) cont.

Eliminate terms with R by subtracting 2nd row from row 1.

From 4th row $\theta_3 = -\frac{2}{\ell}\bar{u}_2$

From 1st row after eliminating R and θ_3

$$\frac{P\ell^3}{EI} = 16.20\,\bar{u}_2 - 4.244\,\bar{v}_2 - 3\ell\,\theta_2$$

$$5^{th}\text{ row}\quad 0 = 12\,\bar{u}_2 - 12\,\bar{v}_2 - 6\ell\,\theta_2 \qquad \times \text{ by } \frac{16.2}{12} \text{ + subtr.}$$

$$\frac{P\ell^3}{EI} = 0 + 11.95\,\bar{v}_2 + 5.1\,\ell\theta_2 \qquad (a)$$

$$3^{rd}\text{ row}\quad 1.5\ell\,\bar{u}_2 + 4.5\ell\,\bar{v}_2 + 5.414\ell^2\theta_2 + \underset{\uparrow\text{elim}}{2\ell^2\theta_3} = 0$$

betw 5th + 3rd rows

$$6\,\bar{v}_2 + 15.66\,\ell\theta_2 = 0$$

$$\therefore\ \bar{v}_2 = -2.61\,\ell\theta_2 \qquad (b)$$

(b) into (a) and add

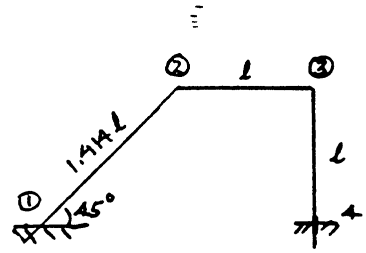
$$\theta_2 = \frac{-1}{26.097}\frac{P\ell^2}{EI} = -.0383\frac{P\ell^2}{EI}$$

$$\bar{v}_2 = .10\,\frac{P\ell^3}{EI}$$

$$\bar{u}_2 = .0808\,\frac{P\ell^3}{EI}$$

$$\theta_3 = -.1616\,\frac{P\ell^2}{EI}$$

10-2) (f)



Element 1-2
$\alpha = 45°$
$c = .707$
$s = .707$

$$\frac{EI}{(1.414\ell)^3}\begin{bmatrix} & \vdots & \\ - - & + & - - \\ & \vdots & \end{bmatrix}\begin{matrix}\bar{u}_2\\ \bar{v}_2\\ \theta_2\end{matrix} = \left(\frac{EI}{\ell^3}\right)\begin{bmatrix} (.1769R+2.122) & (.1769R-2.122) & 1.5\ell \\ (\quad) & (.1769R+2.122) & -1.5\ell \\ 1.5\ell & -1.5\ell & 1.414\ell^2 \end{bmatrix}\begin{matrix}\bar{u}_2\\ \bar{v}_2\\ \theta_2\end{matrix}$$

Element 2-3
$\alpha = 0$
$c = 1$
$s = 0$

$$\left(\frac{EI}{\ell^3}\right)\begin{bmatrix} R & 0 & 0 & -R & 0 & 0 \\ 0 & 12 & 6\ell & 0 & -12 & 6\ell \\ 0 & 6\ell & 4\ell^2 & 0 & -6\ell & 2\ell^2 \\ -R & 0 & 0 & R & 0 & 0 \\ 0 & -12 & -6\ell & 0 & 12 & -6\ell \\ 0 & 6\ell & 2\ell^2 & 0 & -6\ell & 4\ell^2 \end{bmatrix}\begin{matrix}u_2\\ v_2\\ \theta_2\\ u_3\\ v_3\\ \theta_3\end{matrix}$$

Element 3-4
$\alpha = -90°$
$c = 0$
$s = -1$

$$\left(\frac{EI}{\ell^3}\right)\begin{bmatrix} - - - & \vdots & \\ - - - & + & - - - \\ & \vdots & \end{bmatrix}\begin{matrix}u_3\\ v_3\\ \theta_3\end{matrix} = \left(\frac{EI}{\ell^3}\right)\begin{bmatrix} 12 & 0 & 6\ell \\ 0 & R & 0 \\ 6\ell & 0 & 4\ell^2 \end{bmatrix}$$

Assembled stiffness matrix & equation

$$\begin{Bmatrix} P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \left(\frac{EI}{\ell^3}\right)\begin{bmatrix} (1.1769R+2.122) & (.1769R-2.122) & 1.5\ell & -R & 0 & 0 \\ (.1769R-2.122) & (.1769R+14.122) & 4.5\ell & 0 & -12 & 6\ell \\ 1.5\ell & 4.5\ell & 5.414\ell & 0 & -6\ell & 2\ell^2 \\ -R & 0 & 0 & (R+12) & 0 & 6\ell \\ 0 & -12 & -6\ell & 0 & (R+12) & -6\ell \\ 0 & 6\ell & 2\ell^2 & 6\ell & -6\ell & 8\ell^2 \end{bmatrix}\begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ \theta_2 \\ \bar{u}_3 = \bar{u}_2 \\ \bar{v}_3 = 0 \\ \theta_3 \end{Bmatrix}$$

$$\frac{P\ell^3}{EI} = (.1769R+2.122)\bar{u}_2 + (.1769R-2.122)\bar{v}_2 + 1.5\ell\theta_2$$

$$0 = (.1769R-2.122)\bar{u}_2 + (.1769R+2.122)v_2 + 4.5\ell\theta_2 + 6\ell\theta_3$$

$$0 = 1.5\ell\,\bar{u}_2 + 4.5\ell\,\bar{v}_2 + 5.414\ell^2\theta_2 + 2\ell^2\theta_3$$

$$0 = 12\,\bar{u}_2 + 6\ell\,\theta_3$$

$$0 = -12\,\bar{v}_2 - 6\ell\theta_2 - 6\ell\,\theta_3$$

$$0 = 6\ell\,\bar{u}_2 + 6\ell\,v_2 + 2\ell^2\theta_2 + 8\ell^2\theta_3$$

10 - 23 Refer to prob 10-17 with stiffness equation

$$\begin{Bmatrix} F_1 \\ M_1 \\ \hline F_2 \\ M_2 \end{Bmatrix} = \frac{EI}{\ell^3} \left[\begin{array}{cc|cc} 12 & 6\ell & -12 & 6\ell \\ 6 & 4\ell^2 & -6\ell & 2\ell^2 \\ \hline -12 & -6\ell & 12 & -6\ell \\ 6\ell & 2\ell^2 & -6\ell & 4\ell^2 \end{array} \right] \begin{Bmatrix} v_1 \\ \theta_1 \\ \hline v_2 \\ \theta_2 \end{Bmatrix}$$

The moment M_1 at the pin is now altered to $M_1 - K\theta_1$ where K is the torsional spring stiffness at the pin. By transferring the term $-K\theta_1$ into the matrix, the only term altered in the equation is the element (2,2) which becomes $(4\ell^2 + K\frac{\ell^3}{EI})$.

To evalute K, the data specifies the following

$$(mg\ell)\frac{\ell}{2} = K\theta_1 = K\frac{1}{10} \qquad \therefore K = 5mg\ell^2$$

It is also suggested to choose

$$\frac{5mg\ell^3}{EI} = K\frac{\ell}{EI} = 1.0 \qquad \text{or } \frac{K\ell^3}{EI} = 1\,\ell^2$$

Thus the (2,2) term becomes $(4\ell^2 + 1\ell^2) = 5\ell^2$

Whether the problem is solved as a single element or a double element problem, only this one term is changed for the 1st element, which will increase the natural frequencies of the system. Following comparison is given for the single element beam

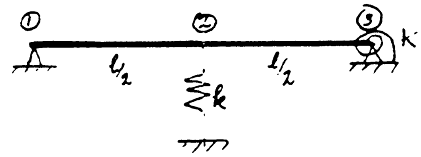
$K = 0 \qquad \omega_1 = 0 \qquad \omega_2 = 17.54\sqrt{\frac{EI}{m\ell^4}}$

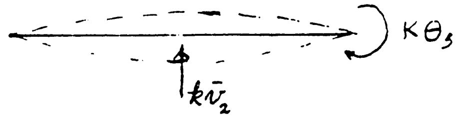
$K\frac{\ell^3}{EI} = 1.0 \qquad \omega_1 = 1.56\sqrt{\frac{EI}{m\ell^4}} \qquad \omega_2 = 19.02\sqrt{\frac{EI}{m\ell^4}}$

10-23 cont

```
»m=[4 13 -3;13 156 -22; -3 -22 4]

m =

     4    13    -3
    13   156   -22
    -3   -22     4

»k=[5 -6 2; -6 12 -6; 2 -6 4]

k =

     5    -6     2
    -6    12    -6
     2    -6     4

»a=inv(m)*k

a =

    5.0000   -9.0000    4.7143
    0.6429   -1.5000    0.9286
    7.7857  -16.5000    9.6429

»eig(a)

ans =

   12.2754
    0.8616
    0.0058
```

10-24 and -25





horiz. displ. = 0

$\bar{u}_1 = \bar{u}_2 = \bar{u}_3 = 0$

Element Stiffness 1-2 and 2-3 Eq. (10.2-1) with $\ell = \frac{\ell}{2}$

$$\frac{8EI}{\ell^3}\begin{bmatrix} 12 & 3\ell & -12 & 3\ell \\ 3\ell & \ell^2 & -3\ell & \frac{1}{2}\ell^2 \\ -12 & -3\ell & \left(12+\frac{k\ell^3}{8EI}\right) & -3\ell \\ 3\ell & \frac{1}{2}\ell^2 & -3\ell & \left(\ell^2+\frac{K\ell^3}{8EI}\right) \end{bmatrix} \quad \text{displ.} = \begin{Bmatrix} \bar{v}_1 \\ \theta_1 \\ \bar{v}_2 \\ \theta_3 \end{Bmatrix} \text{ or } \begin{Bmatrix} \bar{v}_2 \\ \theta_2 \\ \bar{v}_3 \\ \theta_3 \end{Bmatrix}$$

The term $\frac{k\ell^3}{8EI}$ is used only in one of the $\bar{v}_2$ elements

Element Mass 1-2 and 2-3 Eq.(10.2-10) with $\ell = \frac{\ell}{2}$

$$\frac{m\ell}{840}\begin{bmatrix} 156 & 11\ell & 54 & -6.5\ell \\ 11\ell & \ell^2 & 6.5\ell & -.75\ell^2 \\ 54 & 6.5\ell & 156 & -11\ell \\ -6.5\ell & -.75\ell^2 & -11\ell & \ell^2 \end{bmatrix}$$

Assembled

$$\frac{8EI}{\ell^3}\begin{bmatrix} 12 & 3\ell & -12 & 3\ell & & \\ 3\ell & \ell^2 & -3\ell & .5\ell^2 & & \\ -12 & -3\ell & \left(24+\frac{k\ell^3}{8EI}\right) & 0 & -12 & 3\ell \\ 3\ell & .5\ell^2 & 0 & 2\ell^2 & -3\ell & .5\ell^2 \\ & & -12 & -3\ell & 12 & -3\ell \\ & & 3\ell & .5\ell^2 & -3\ell & \left(\ell^2+\frac{K\ell^3}{8EI}\right) \end{bmatrix} \quad \text{displ} = \begin{Bmatrix} \bar{v}_1 = 0 \\ \theta_1 \\ \bar{v}_2 \\ \theta_2 \\ \bar{v}_3 = 0 \\ \theta_3 \end{Bmatrix}$$

$\therefore$ delete cols 1 and 5 and rows 1 and 5

10 – 24 – 25 Cont.

Assembled.

$$\frac{ml}{840}\begin{bmatrix} 156 & 11l & 54 & -6.5l & & \\ 11l & l^2 & 6.5l & -.75l^2 & & \\ 54 & 6.5l & 312 & 0 & 54 & -6.5l \\ -6.5l & -.75l^2 & 0 & 2l^2 & 6.5l & -.75l^2 \\ & & 54 & 6.5l & 156 & -11l \\ & & -6.5l & -.75l^2 & -11l & l^2 \end{bmatrix}$$

Eq. of motion is a 4×4

$$\left[-\frac{\omega^2 ml}{840}\begin{bmatrix} l^2 & 6.5l & -.75l^2 & 0 \\ 6.5l & 312 & 0 & -6.5l \\ -.75l^2 & 0 & 2l^2 & -.75l^2 \\ 0 & -6.5l & -.75l^2 & l^2 \end{bmatrix} + \frac{8EI}{l^3}\begin{bmatrix} l^2 & -3l & .5l^2 & 0 \\ -3l & \left(24+\frac{kl^3}{8EI}\right) & 0 & 3l \\ .5l^2 & 0 & 2l^2 & .5l^2 \\ 0 & 3l & .5l^2 & \left(l^2+\frac{Kl^3}{8EI}\right) \end{bmatrix}\right]\begin{Bmatrix} \theta_1 \\ v_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\left.\begin{aligned} \frac{kl^3}{EI} &= \frac{1}{2} \\ \frac{Kl^3}{EI} &= \frac{1}{4}l^2 \end{aligned}\right\}$$ for computer solution let $l = 1$ inside the matrices

$$\lambda = \frac{\omega^2 m l^4}{6720\,EI} \qquad \omega = \sqrt{6720\lambda}\sqrt{\frac{EI}{ml^4}}$$

10 - 25 cont

with springs

```
»m

m =

    1.0000    6.5000   -0.7500         0
    6.5000  312.0000         0   -6.5000
   -0.7500         0    2.0000   -0.7500
         0   -6.5000   -0.7500    1.0000

»k

k =

    1.0000   -3.0000    0.5000         0
   -3.0000   24.0600         0    3.0000
    0.5000         0    2.0000    0.5000
         0    3.0000    0.5000    1.0310

»a

a =

    1.0000   -0.5784    1.2239    0.5916
   -0.5784    0.3904   -0.6436   -0.0179
    1.2239   -0.6436    2.7582    2.2054
    0.5916   -0.0179    2.2054    4.0139

»[v,d]=eig(a)

v =

    0.2478    0.5229    0.6611   -0.4776
   -0.0977   -0.4054   -0.2697   -0.8680
    0.6131    0.4508   -0.6439   -0.0795
    0.7438   -0.5991    0.2750    0.1106
```

$$\omega = \sqrt{6720\,\lambda}\,\sqrt{\frac{EI}{m\ell^4}}$$

```
d =

    6.0312         0         0         0
         0    1.8257         0         0
         0         0    0.2901         0
         0         0         0    0.0155
```

$\omega =$ 10.19
44.15
110.8
201.3

10-25 cont

```
»m

m =

    1.0000    6.5000   -0.7500         0
    6.5000  312.0000         0   -6.5000
   -0.7500         0    2.0000   -0.7500
         0   -6.5000   -0.7500    1.0000
```

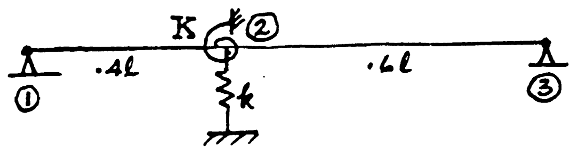
Without Springs.

```
k =

    1.0000   -3.0000    0.5000         0
   -3.0000   24.0000         0    3.0000
    0.5000         0    2.0000    0.5000
         0    3.0000    0.5000    1.0000

»a=qin'*k*qin

a =

    1.0000   -0.5784    1.2239    0.5916
   -0.5784    0.3902   -0.6435   -0.0180
    1.2239   -0.6435    2.7582    2.2055
    0.5916   -0.0180    2.2055    3.9571

»[v,d]=eig(a)

v =

    0.2500    0.5246    0.6614   -0.4742
   -0.0989   -0.4082   -0.2618   -0.8689
    0.6171    0.4430   -0.6449   -0.0841
    0.7395   -0.6016    0.2795    0.1142

d =

    6.0000         0         0         0
         0    1.8052         0         0
         0         0    0.2857         0
         0         0         0    0.0146
```

$\omega = \sqrt{6720\lambda}\sqrt{\frac{EI}{m\ell^4}}$

		exact
$\omega =$	9.908	9.87
	43.817	39.48
	110.13	88.5
	200.8	156

10-25



Stiffness Matrix

Element ① to ② $\ell_1 = .4\ell$, $v_1 = 0$

$$\frac{EI}{\ell_1^3}\begin{bmatrix} 4\ell_1^2 & -6\ell_1 & 2\ell_1^2 \\ -6\ell_1 & 12 & -6\ell_1 \\ 2\ell_1^2 & -6\ell_1 & 4\ell_1^2 \end{bmatrix}\begin{Bmatrix}\theta_1 \\ v_2 \\ \theta_2\end{Bmatrix} = \frac{EI}{.064\ell^3}\begin{bmatrix} 4(.16\ell^2) & -6(.4\ell) & 2(.16\ell^2) \\ -6(.4\ell) & 12 & -6(.4\ell) \\ 2(.16\ell^2) & -6(.4\ell) & 4(.16\ell^2) \end{bmatrix}$$

(Let $\ell = 1$.)

$$= \frac{EI}{\ell^3}\begin{bmatrix} 10 & -37.5 & 5 \\ -37.5 & 187.5 & -37.5 \\ 5 & -37.5 & 10 \end{bmatrix}\begin{Bmatrix}\theta_1 \\ v_2 \\ \theta_2\end{Bmatrix}$$

Element ② to ③ $\ell_2 = .6\ell$, $v_3 = 0$

$$\frac{EI}{\ell_2^3}\begin{vmatrix} 12 & 6\ell_2 & 6\ell_2 \\ 6\ell_2 & 4\ell_2^2 & 2\ell_2^2 \\ 6\ell_2 & 2\ell_2^2 & 4\ell_2^2 \end{vmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \\ \theta_3\end{Bmatrix} = \frac{EI}{.216\ell^3}\begin{bmatrix} 12 & 6(.6\ell) & 6(.6\ell) \\ 6(.6\ell) & 4(.36\ell^2) & 2(.36\ell^2) \\ 6(.6\ell) & 2(.36\ell^2) & 4(.36\ell^2) \end{bmatrix}$$

$$= \frac{EI}{\ell^3}\begin{bmatrix} 55.56 & 16.67 & 16.67 \\ 16.67 & 6.67 & 3.33 \\ 16.67 & 3.33 & 6.67 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \\ \theta_3\end{Bmatrix}$$

Assemble

$$\frac{EI}{\ell^3}\begin{bmatrix} 10 & -37.5 & 5 & 0 \\ -37.5 & \begin{pmatrix}187.5 \\ 55.56\end{pmatrix} & \begin{pmatrix}-37.5 \\ 16.67\end{pmatrix} & 16.67 \\ 5 & \begin{pmatrix}-37.5 \\ 16.67\end{pmatrix} & \begin{pmatrix}10 \\ 6.67\end{pmatrix} & 3.33 \\ 0 & 16.67 & 3.33 & 6.67 \end{bmatrix}\begin{Bmatrix}\theta_1 \\ v_2 \\ \theta_2 \\ \theta_3\end{Bmatrix}$$

$$= \frac{EI}{\ell^3}\begin{bmatrix} 10 & -37.5 & 5 & 0 \\ -37.5 & 243.1 & -20.83 & 16.67 \\ 5 & -20.83 & 16.67 & 3.33 \\ 0 & 16.67 & 3.33 & 6.67 \end{bmatrix}\begin{Bmatrix}\theta_1 \\ v_2 \\ \theta_2 \\ \theta_3\end{Bmatrix}$$

must add $\frac{k\ell^3}{EI} = 1$ to $(2,2)$ and $\frac{K\ell^3}{EI} = 2$ to $(3,3)$

10-25 Cont.

Mass Matrix

Element ① to ② $l_1 = .4l$, $v_1 = 0$

$$\frac{ml_1}{420}\begin{bmatrix} 4l_1^2 & 13l_1 & -3l^2 \\ 13l_1 & 156 & -22l_1 \\ -3l_1^2 & -22l_1 & 4l_1^2 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \frac{m.4l}{420}\begin{bmatrix} 4(.16) & 13(.4) & -3(.16) \\ 13(.4) & 156 & -22(.4) \\ -3(.16) & -22(.4) & 4(.16) \end{bmatrix}$$

$$= \frac{ml}{420}\begin{bmatrix} .256 & 2.08 & -.192 \\ 2.08 & 62.4 & -3.52 \\ -.192 & -3.52 & .256 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

Element ② to ③ $l_2 = .6l$, $v_3 = 0$

$$.6\frac{ml}{420}\begin{bmatrix} 156 & 22l_2 & -13l_2 \\ 22l_2 & 4l_2^2 & -3l_2^2 \\ -13l_2 & -3l_2^2 & 4l_2^2 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \frac{ml}{420}\begin{bmatrix} 93.6 & 7.92 & -4.68 \\ 7.92 & .864 & -.648 \\ -4.68 & -.648 & .864 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix}$$

Assemble

$$\frac{ml}{420}\begin{bmatrix} .256 & 2.08 & -.192 & 0 \\ 2.08 & \begin{pmatrix} 62.4 \\ 93.6 \end{pmatrix} & \begin{pmatrix} -3.52 \\ 7.92 \end{pmatrix} & -4.68 \\ -.192 & \begin{pmatrix} -3.52 \\ 7.92 \end{pmatrix} & \begin{pmatrix} .256 \\ .864 \end{pmatrix} & -.648 \\ 0 & -4.68 & -.648 & .864 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ v_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix}$$

$$= \frac{ml}{420}\begin{bmatrix} .256 & 2.08 & -.192 & 0 \\ 2.08 & 156 & 4.40 & -4.68 \\ -.192 & 4.40 & 1.20 & -.648 \\ 0 & -4.68 & -.648 & .864 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ v_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix}$$

Eq. of motion

$$-\lambda\begin{bmatrix} .256 & 2.08 & -.192 & 0 \\ 2.08 & 156. & 4.40 & -4.68 \\ -.192 & 4.40 & 1.20 & -.648 \\ 0 & -4.68 & -.648 & .864 \end{bmatrix} + \begin{bmatrix} 10 & -37.5 & 5 & 0 \\ -37.5 & 243.1 & -20.83 & 16.67 \\ 5 & -20.83 & 16.67 & 3.33 \\ 0 & 16.67 & 3.33 & 6.67 \end{bmatrix}$$

10-26 cont

```
»m=[.2560 2.08 -.192 0; 2.08 156 4.4 -4.68;-.192 4.4 1.12 -.648;
0 -4.68 -6.48 .864]

m =

    0.2560    2.0800   -0.1920         0
    2.0800  156.0000    4.4000   -4.6800
   -0.1920    4.4000    1.1200   -0.6480
         0   -4.6800   -6.4800    0.8640

»k=[10 -37.5 5 0; -37.5 243.1 -20.83 16.67; 5 -20.83 16.67 3.333;
0 16.67 3.333 6.67]

k =
```

without springs

```
   10.0000  -37.5000    5.0000         0
  -37.5000  243.1000  -20.8300   16.6700
    5.0000  -20.8300   16.6700    3.3330
         0   16.6700    3.3330    6.6700

»q=chol(m)

q =

    0.5060    4.1110   -0.3795         0
         0   11.7941    0.5053   -0.3968
         0         0    0.8489   -0.5271
         0         0         0    0.6547

»qin=inv(q)

qin =

    1.9764   -0.6889    1.2936    0.6239
         0    0.0848   -0.0505    0.0108
         0         0    1.1780    0.9484
         0         0         0    1.5273

»a=qin'*k*qin

a =

   39.0625  -19.8999   40.9488   20.9071
  -19.8999   10.8744  -21.5071   -8.5659
```

10-26 cont

40.9488	-21.5071	63.0979	42.4767
20.9071	-8.5659	42.4767	49.6672

```
»[v,d]=eig(a)

v =

   -0.4726   -0.5505   -0.5870   -0.3591
    0.2374    0.3491    0.0356   -0.9058
   -0.6816   -0.0576    0.7086   -0.1730
   -0.5056    0.7561   -0.3899    0.1436
```

$\lambda = \frac{\omega^2 m l^4}{420 EI}$ $\quad \omega = \sqrt{420\lambda}\sqrt{\frac{EI}{ml^4}}$

```
d =

  130.4892         0         0         0
         0   27.2528         0         0
         0         0    4.7246         0
         0         0         0    0.2353
```

$\omega =$ 234
106.9
44.55
9.943

with springs

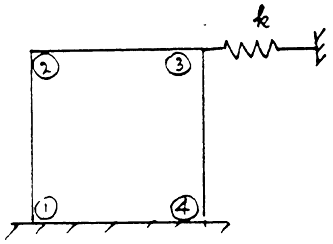
```
k =

   10.0000  -37.5000    5.0000         0
  -37.5000  244.1000  -20.8300   16.6700
    5.0000  -20.8300   18.6700    3.3330
         0   16.6700    3.3330    6.6700

»a=qin'*k*qin

a =

   39.0625  -19.8999   40.9488   20.9071
  -19.8999   10.8816  -21.5114   -8.5650
   40.9488  -21.5114   65.8757   44.7105
   20.9071   -8.5650   44.7105   51.4662
```

10-26 cont

```
»[v,d]=eig(a)

v =

   -0.4588    0.5716   -0.5727    0.3670
    0.2303   -0.3566    0.0393    0.9046
   -0.6875    0.0499    0.7057    0.1641
   -0.5137   -0.7373   -0.4153   -0.1418


d =

  133.8189         0         0         0
         0   28.0864         0         0
         0         0    5.1330         0
         0         0         0    0.2477
```

$\omega =$ 237
108.6
44.44
10.20

10-27



Stiffness matrix

$$\frac{EI}{\ell^3}\begin{bmatrix} (12+R) & 0 & 6\ell & -R & 0 & 0 \\ \cancel{0} & \cancel{(12+R)} & \cancel{6\ell} & \cancel{0} & \cancel{-12} & \cancel{6\ell} \\ 6\ell & 6\ell & 8\ell^2 & 0 & -6\ell & 2\ell^2 \\ -R & 0 & 0 & \left(12+R+\frac{k\ell^3}{EI}\right) & 0 & 6\ell \\ \cancel{0} & \cancel{-12} & \cancel{-6\ell} & \cancel{0} & \cancel{(12+R)} & \cancel{-6\ell} \\ 0 & 6\ell & 2\ell^2 & 6\ell & -6\ell & 8\ell^2 \end{bmatrix} \begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 = 0 \\ \theta_2 \\ \bar{u}_3 = \bar{u}_2 \\ \bar{v}_3 = 0 \\ \theta_3 \end{Bmatrix}$$

Upper left qtr = lower right qtr of elem. 1-2 plus upper left qtr of elem. 2-3.

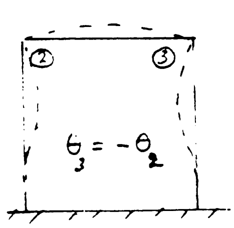
Lower right qtr = upper left qtr of elem. 3-4 plus lower right qtr of elem. 2-3.

mass matrix

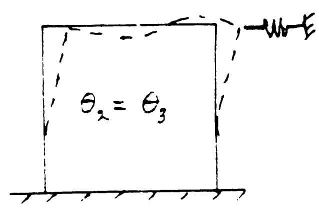
$$\frac{m\ell}{420}\begin{bmatrix} (156+N) & 0 & 22\ell & \frac{1}{2}N & 0 & 0 \\ \cancel{0} & \cancel{(156+N)} & \cancel{22\ell} & \cancel{0} & \cancel{54} & \cancel{-13\ell} \\ 22\ell & 22\ell & 8\ell^2 & 0 & 13\ell & -3\ell^2 \\ \frac{1}{2}N & 0 & 0 & (156+N) & 0 & 22\ell \\ \cancel{0} & \cancel{54} & \cancel{-13\ell} & \cancel{0} & \cancel{(156+N)} & \cancel{-22\ell} \\ 0 & -13\ell & -3\ell^2 & 22\ell & -22\ell & 8\ell^2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_2 \\ \ddot{v}_2 \\ \ddot{\theta}_2 \\ \ddot{u}_3 \\ \ddot{v}_3 \\ \ddot{\theta}_3 \end{Bmatrix}$$

Assumptions

$\bar{u}_2 = \bar{u}_3$, $\bar{v}_2 = \bar{v}_3 = 0$, two modes $\begin{cases} \theta_3 = -\theta_2 & \rightarrow \text{spring not involved} \\ \theta_3 = \theta_2 \end{cases}$



Spring not in action



10-27 Cont

General eq. of motion

$$\frac{m\ell}{420}\begin{bmatrix}(156+N) & 22\ell & \frac{1}{2}N & 0\\ 22\ell & 8\ell^2 & 0 & -3\ell^2\\ \frac{1}{2}N & 0 & (156+N) & 22\ell\\ 0 & -3\ell^2 & 22\ell & 8\ell^2\end{bmatrix}\begin{Bmatrix}\ddot{\bar{u}}_2\\ \ddot{\theta}_2\\ \ddot{\bar{u}}_3\\ \ddot{\theta}_3\end{Bmatrix} + \frac{EI}{\ell^3}\begin{bmatrix}(12+R) & 6\ell & -R & 0\\ 6\ell & 8\ell^2 & 0 & 2\ell^2\\ -R & 0 & (12+R+\frac{k\ell^3}{EI}) & 6\ell\\ 0 & 2\ell^2 & 6\ell & 8\ell^2\end{bmatrix}\begin{Bmatrix}\bar{u}_2\\ \theta_2\\ \bar{u}_3\\ \theta_3\end{Bmatrix} = \begin{Bmatrix}0\\0\\0\\0\end{Bmatrix}$$

Symmetric mode $\bar{u}_2 = \bar{u}_3 = 0$ $\therefore$ cross out col. 1, col. 3

$\theta_3 = -\theta_2$ row 1, row 3

$$\frac{m\ell}{420}\begin{bmatrix}8\ell^2 & -3\ell^2\\ -3\ell & 8\ell^2\end{bmatrix}\begin{Bmatrix}\ddot{\theta}_2\\ \ddot{\theta}_3\end{Bmatrix} + \frac{EI}{\ell^3}\begin{bmatrix}8\ell^2 & 2\ell^2\\ 2\ell^2 & 8\ell^2\end{bmatrix}\begin{Bmatrix}\theta_2\\ \theta_3\end{Bmatrix} = \begin{Bmatrix}0\\0\end{Bmatrix}$$

$$\left[-\omega^2\frac{m\ell}{420}(11) + \frac{EI}{\ell^3}(6)\right]\ell^2\theta_2 = 0$$

$$\omega^2 = \frac{6\times 420}{11}\frac{EI}{m\ell^4} \qquad \underline{\underline{\omega = 15.14\sqrt{\frac{EI}{m\ell^4}}}}$$

Antisym. mode $\bar{u}_2 = \bar{u}_3 \neq 0$, $\theta_2 = \theta_3$

$$\left[-\frac{\omega^2 m\ell}{420}\begin{bmatrix}(156+\frac{3}{2}N) & 22\ell\\ 22\ell & 5\ell^2\end{bmatrix} + \frac{EI}{\ell^3}\begin{bmatrix}(12+\frac{k\ell^3}{EI}) & 6\ell\\ 6\ell & 10\ell^2\end{bmatrix}\right]\begin{Bmatrix}\bar{u}\\ \theta\end{Bmatrix} = \begin{Bmatrix}0\\0\end{Bmatrix}$$

with $\lambda = \left(\frac{\omega^2 m\ell^4}{420EI}\right)$ $N = 140$ Characteristic eq. =

$$\begin{vmatrix}(12+\frac{k\ell^3}{EI}-366\lambda) & (6-22\lambda)\ell\\ (6-22\lambda)\ell & (10-5\lambda)\ell^2\end{vmatrix} = 0$$

$$\lambda^2 - \left(2.567 + .003715\frac{k\ell^3}{EI}\right)\lambda + .062407 = 0$$

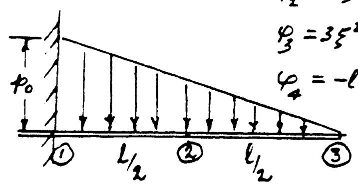
$\therefore$ must decide on numerical value of $\frac{k\ell^3}{EI}$ before solving

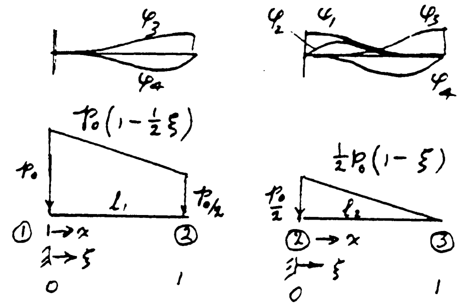
If $\frac{k\ell^3}{EI} = 0$, then $\lambda_1 = .02454$ $\omega_1 = 3.21\sqrt{EI/m\ell^4}$

$\lambda_2 = 2.543$ $\omega_2 = 32.63\sqrt{EI/m\ell^4}$

10 - 28 (a)

$$\varphi_1 = 1-3\xi^2+2\xi^3$$
$$\varphi_2 = l\xi - 2l\xi^2 + l\xi^3$$
$$\varphi_3 = 3\xi^2 - 2\xi^3$$
$$\varphi_4 = -l\xi^2 + l\xi^3$$





Element ①-② See Eq.(10.7-4)

since $\bar{v}_1 = 0$ only φ_3 & φ_4 are needed.

$$\bar{F}_2 = \int_0^1 p_0(1-\tfrac{1}{2}\xi)(3\xi^2-2\xi^3)l_1\,d\xi$$

$$M_2 = \int_0^1 p_0(1-\tfrac{1}{2}\xi)(-l\xi^2+l\xi^3)l_1\,d\xi$$

Integrated results.

$$\bar{F}_2 = .3250\, p_0 l_1$$

$$M_2 = -.0583\, p_0 l_1^2$$

Element ②-③ See Eq.(10.7-4)

all 4 φ_s are necessary

$$\bar{F}_3 = \int_0^1 \frac{p_0}{2}(1-\xi)(3\xi^2-2\xi^3)l_2\,d\xi$$

$$M_3 = \int_0^1 \frac{p_0}{2}(1-\xi)(-l_2\xi^2+l\xi^3)l_2\,d\xi$$

$$\bar{F}_2 = -\int_0^1 \frac{p_0}{2}(1-\xi)(1-3\xi^2+2\xi^3)\,l_2\,d\xi$$

$$M_2 = -\int_0^1 \frac{p_0}{2}(1-\xi)(l_2\xi-2l\xi^2+l\xi^3)l_2\,d\xi$$

Integrated results

$$\bar{F}_2 = .25\, p_0 l_2$$

$$M_2 = -.850\, p_0 l_2^2$$

$$\bar{F}_3 = .0750\, p_0 l_2$$

$$M_3 = .0583\, p_0 l_2^2$$

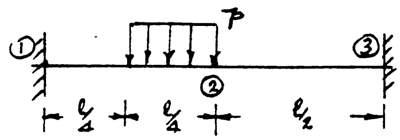
Assembly of both sections

$$\begin{Bmatrix}\bar{F}_2\\ M_2\\ \bar{F}_3\\ M_3\end{Bmatrix} = -p_0\begin{Bmatrix} .325\,l_1 + .250\,l_2\\ -.0583\,l_1^2 - .850\,l_2^2\\ .0750\,l_2\\ .0583\,l_2^2\end{Bmatrix}$$

with $l_1 = l_2 = l/2$

$$\begin{Bmatrix}\bar{F}_2\\ M_2\\ \bar{F}_3\\ M_3\end{Bmatrix} = -p_0 l\begin{Bmatrix} .2875\\ -.2271\,l\\ .0375\\ .0146\,l\end{Bmatrix}$$

10-28(b)



$$\bar{F}_2 = -p\int_{1/2}^{1} \varphi_3\, l\, d\xi = -p\int_{1/2}^{1}(3\xi^2-2\xi^3)\ell_1 d\xi = -p\ell_1\left[3\left(\frac{\xi^3}{3}\right)-2\left(\frac{\xi^4}{4}\right)\right]_{1/2}^{1}$$

$$= -p\left[.4063\right]\ell_1 = -p\ell\left[.2031\right] \qquad \ell_1 = \frac{\ell}{2}$$

$$M_2 = -p\int_{1/2}^{1}(-\ell_1\xi^2+\ell_1\xi^3)\ell_1 d\xi \quad = .0286\ p\ell^2$$

with $\bar{v}_1 = \bar{v}_3 = \theta_1 = \theta_3 = 0$ (see Ex. 10.6-1)

$$\begin{Bmatrix} -P_2 \\ M_2 \end{Bmatrix} = \frac{EI}{\ell^3}\begin{bmatrix} 192 & \\ & 16\ell^2 \end{bmatrix}\begin{Bmatrix} \bar{v}_2 \\ \theta_2 \end{Bmatrix} = -p\ell\begin{Bmatrix} .2031 \\ .0286\ell \end{Bmatrix}$$

$$\therefore\ \bar{v}_2 = -\frac{.2031}{192}\frac{p\ell^4}{EI}$$

$$\theta_2 = -\frac{.0286}{16}\frac{p\ell^3}{EI}$$

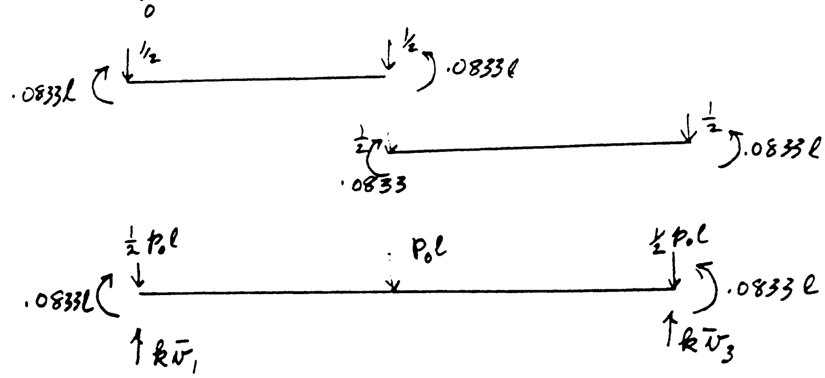
10-29 Cont.

$$\bar{F}_1 = -p_0 f(t) \int_0^1 \varphi_1 \, l \, d\xi = -p_0 f(t) \left[1 - 1 + \tfrac{1}{2} \right] l = -\tfrac{1}{2} \, p_0 \, l \, f(t)$$

$$\bar{M}_1 = -p_0 f(t) \, l \int_0^1 \varphi_2 \, l \, d\xi = -p_0 \, l^2 f(t) \left[\tfrac{1}{2} - \tfrac{2}{3} + \tfrac{1}{4} \right] = -.0833 \, p_0 \, l^2 f(t)$$

$$\bar{F}_2 = -p_0 f(t) \int_0^1 \varphi_3 \, l \, d\xi = -p_0 \, l \, f(t) \left[1 - \tfrac{2}{4} \right] = -\tfrac{1}{2} \, p_0 \, l \, f(t)$$

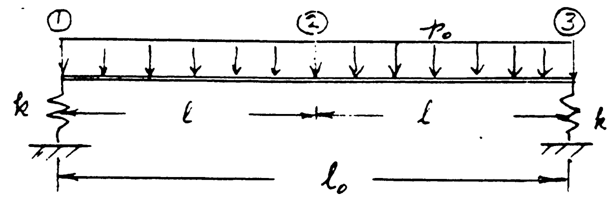
$$\bar{M}_2 = -p_0 f(t) \, l \int_0^1 \varphi_4 \, l \, d\xi = -p_0 \, l^2 f(t) \left[-\tfrac{1}{3} + \tfrac{1}{4} \right] = .0833 \, p_0 \, l^2 f(t)$$



$$\begin{Bmatrix} \bar{F}_1 - k\bar{v}_1 \\ \bar{M}_1 \\ \bar{F}_2 \\ \bar{M}_2 \\ \bar{F}_3 - k\bar{v}_3 \\ M_3 \end{Bmatrix} = \begin{Bmatrix} -.50 - k\bar{v}_1 \\ -.0833 \, l \\ -1.0 \\ 0 \\ -.50 - k\bar{v}_3 \\ .0833 \, l \end{Bmatrix} p_0 \, l \, f(t)$$

$-k\bar{v}_1$ and $-k\bar{v}_3$ must now be shifted to stiffness matrix as $+k\bar{v}_1$ and $+k\bar{v}_3$

10 - 29



Element $\begin{pmatrix} 1-2 \\ 2-3 \end{pmatrix}$ Eq.(10.2-1) for $\bar{k}$ and Eq.(10.2-10) for $\bar{m}$

$$\bar{k} = \frac{EI}{\ell^3}\begin{bmatrix} 12 & 6\ell & -12 & 6\ell \\ 6\ell & 4\ell^2 & -6\ell & 2\ell^2 \\ -12 & -6\ell & 12 & -6\ell \\ 6\ell & 2\ell^2 & -6\ell & 4\ell^2 \end{bmatrix} \qquad \bar{m} = \frac{m\ell}{420}\begin{bmatrix} 156 & 22\ell & 54 & -13\ell \\ 22\ell & 4\ell^2 & 13\ell & -3\ell^2 \\ 54 & 13\ell & 156 & -22\ell \\ -13\ell & -3\ell^2 & -22\ell & 4\ell^2 \end{bmatrix}$$

Assembled

$$\bar{k} = \frac{EI}{\ell^3}\begin{bmatrix} 12 & 6\ell & -12 & 6\ell & & \\ 6\ell & 4\ell^2 & -6\ell & 2\ell^2 & & \\ -12 & -6\ell & 24 & 0 & -12 & 6\ell \\ 6\ell & 2\ell^2 & 0 & 8\ell^2 & -6\ell & 2\ell^2 \\ & & -12 & -6\ell & 12 & -6\ell \\ & & 6\ell & 2\ell^2 & -6\ell & 4\ell^2 \end{bmatrix}\begin{Bmatrix} \bar{v}_1 \\ \theta_1 \\ \bar{v}_2 \\ \theta_2 \\ \bar{v}_3 \\ \theta_3 \end{Bmatrix}$$

must replace ℓ by $\frac{\ell_0}{2}$

$$\bar{m} = \frac{m\ell}{420}\begin{bmatrix} 156 & 22\ell & 54 & -13\ell & & \\ 22\ell & 4\ell^2 & 13\ell & -3\ell^2 & & \\ 54 & 13\ell & 312 & 0 & 54 & -13\ell \\ -13\ell & -3\ell^2 & 0 & 8\ell^2 & 13\ell & -3\ell^2 \\ & & 54 & 13\ell & 156 & -22\ell \\ & & -13\ell & -3\ell^2 & -22\ell & 4\ell^2 \end{bmatrix}$$

For loads, must determine generalized forces & moments at stations

10-29 Cont. General equation of motion 6×6

$$\frac{m\ell}{420}\begin{bmatrix} 156 & 22\ell & 54 & -13\ell & 0 & 0 \\ 22\ell & 4\ell^2 & 13\ell & -3\ell^2 & 0 & 0 \\ 54 & 13\ell & 312 & 0 & 54 & -13\ell \\ -13\ell & -3\ell & 0 & 8\ell^2 & 13\ell & -3\ell^2 \\ 0 & 0 & 54 & 13\ell & 156 & -22\ell \\ 0 & 0 & -13\ell & -3\ell^2 & -22\ell & 4\ell^2 \end{bmatrix}\begin{Bmatrix} \ddot{\bar{v}}_1 \\ \ddot{\theta}_1 \\ \ddot{\bar{v}}_2 \\ \ddot{\theta}_2 \\ \ddot{\bar{v}}_3 \\ \ddot{\theta}_3 \end{Bmatrix} + \frac{EI}{\ell^3}\begin{bmatrix} (12+\frac{k\ell^3}{EI}) & 6\ell & -12 & 6\ell & 0 & 0 \\ 6\ell & 4\ell^2 & -6\ell & 2\ell^2 & 0 & 0 \\ -12 & -6\ell & 24 & 0 & -12 & 6\ell \\ 6\ell & 2\ell^2 & 0 & 8\ell^2 & -6\ell & 2\ell^2 \\ 0 & 0 & -12 & -6\ell & (12+\frac{k\ell^3}{EI}) & -6\ell \\ 0 & 0 & 6\ell & 2\ell^2 & -6\ell & 4\ell^2 \end{bmatrix}\begin{Bmatrix} \bar{v}_1 \\ \theta_1 \\ \bar{v}_2 \\ \theta_2 \\ \bar{v}_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} -.50 \\ -.0833\ell \\ -1.0 \\ 0 \\ -.50 \\ .0833\ell \end{Bmatrix} p_0 \ell f(t)$$

10-30

Symmetric mode (free-free vibration)

$$\bar{v}_1 = \bar{v}_3 , \qquad \theta_2 = 0 \qquad \theta_1 = -\theta_3$$

Due to symmetry only $\bar{v}_1$ θ_1 and $\bar{v}_2$ are necessary

From above equation we have

$$\frac{m\ell}{420}\begin{bmatrix} 156 & 22\ell & 54 \\ 22\ell & 4\ell^2 & 13\ell \\ 54 & 13\ell & 312 \end{bmatrix}\begin{Bmatrix} \ddot{\bar{v}}_1 \\ \ddot{\theta}_1 \\ \ddot{\bar{v}}_2 \end{Bmatrix} + \frac{EI}{\ell^3}\begin{bmatrix} (12+\frac{k\ell^3}{EI}) & 6\ell & -12 \\ 6\ell & 4\ell^2 & -6\ell \\ -12 & -6\ell & 24 \end{bmatrix}\begin{Bmatrix} \bar{v}_1 \\ \theta_1 \\ \bar{v}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Let. $\frac{k\ell^3}{EI} = 2$

Computer solution follows:

With $\frac{k\ell^3}{EI} = 2$, the end springs are very flexible and the beam behaves almost like a rigid bar, as shown by the first mode

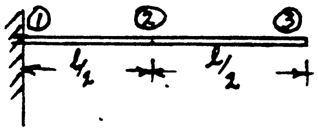
10-30 cont

```
m =

   156    22    54
    22     4    13
    54    13   312


k =

    14     6   -12
     6     4    -6
   -12    -6    24

»a=qin'*k*qin
```

qin = inv (chol (m))

```
a =

    0.0897    0.3402   -0.2032
    0.3402    2.8817   -1.3864
   -0.2032   -1.3864    0.7490

»[v,d]=eig(a)

v =

    0.8052   -0.1116   -0.5824
    0.1784   -0.8911    0.4173
    0.5656    0.4399    0.6976


d =

    0.0224         0         0
         0    3.6087         0
         0         0    0.0893
```

10-31 Eq. of motion for Example 10.5-1 is



$$\left[-\lambda \begin{bmatrix} 312 & 0 & 54 & -6.5l \\ 0 & 2l^2 & 6.5l & -.75l^2 \\ 54 & 6.5l & 156 & -11l \\ -6.5l & -.75l^2 & -11l & l^2 \end{bmatrix} + \begin{bmatrix} 24 & 0 & -12 & 3l \\ 0 & 2l^2 & -3l & .5l^2 \\ -12 & -3l & 12 & -3l \\ 3l & .5l & -3l & l^2 \end{bmatrix} \right] \begin{Bmatrix} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\lambda = \left(\frac{\omega^2 m l}{840}\right)\left(\frac{l^3}{8EI}\right) = \frac{\omega^2 m l^4}{6720\, EI}$$

where $\frac{l}{2}$ has been substituted for l.

Since l inside of matrices factor out of the equation

Let $l = 1.0$ inside of the matrices

Input into computer program with $l = 1$.

Solution gives

$$\omega_1 = 3.517 \sqrt{\frac{EI}{ml^4}}$$

$$\omega_2 = 22.22 \quad ''$$

$$\omega_3 = 75.16 \quad ''$$

$$\omega_4 = 218.1 \quad ''$$

10-31 cont

m =

312.0000	0	54.0000	-6. 000
0	2.0000	6.5000	-0.7500
54.0000	6.5000	156.0000	-11.0000
-6.5000	-0.7500	-11.0000	1.0000

»k

k =

24.0000	0	-12.0000	3.0000
0	2.0000	-3.0000	0.5000
-12.0000	-3.0000	12.0000	-3.0000
3.0000	0.5000	-3.0000	1.0000

a =

0.0769	0	-0.0816	0.3811
0	1.0000	-0.5996	1.2864
-0.0816	-0.5996	0.4580	-1.3319
0.3811	1.2864	-1.3319	6.4619

»[v,d]=eig(a)

v =

-0.0542	0.1011	0.5904	-0.7989
-0.2219	-0.8784	0.3807	0.1852
0.2119	0.3722	0.7108	0.5580
-0.9502	0.2823	0.0359	0.1267

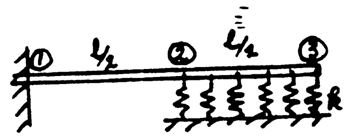
$$\lambda = \frac{\omega^2 m \ell^4}{6720\, EI}$$

d =

7.0810	0	0	0
0	0.8406	0	0
0	0	0.0018	0
0	0	0	0.0735

ω = 218.13
75.16
3.517
22.22

10 – 32



See Ex. 10.8–1

For 2 element beam, the form of the equation of motion is

$$\frac{m\ell}{840}\left[m_{ij}\right]\begin{Bmatrix}\ddot{v}_2\\ \ddot{\theta}_2\\ \ddot{v}_3\\ \ddot{\theta}_3\end{Bmatrix} + \frac{8EI}{\ell^3}\left[k_{ij}\right]\begin{Bmatrix}v_2\\ \theta_2\\ v_3\\ \theta_3\end{Bmatrix} + \frac{8EI}{\ell^3}\left(\frac{k\ell^3}{8EI}\right)\left[q_{ij}\right]\begin{Bmatrix}v_2\\ \theta_2\\ v_3\\ \theta_3\end{Bmatrix} = \begin{Bmatrix}0\\0\\0\\0\end{Bmatrix}$$

where $\{Q_i\}$ in text is modified for $\ell = \frac{\ell}{2}$ and $\frac{k\ell^3}{8EI} = 10$.

$$\{Q_i\} = -k\begin{bmatrix} .1857\ell & .01311\ell^2 & .0643\ell & -.007736\ell^2 \\ & .001191\ell^3 & .007738\ell^2 & -.000893\ell^3 \\ & & .1857\ell & -.013075\ell^2 \\ & & & .001191\ell^3 \end{bmatrix} \times 10.$$

with $\lambda = \frac{\omega^2 m\ell^4}{6720\,EI}$

$$\left[-\lambda\left[m_{ij}\right] + \left[k_{ij} + \frac{k\ell^3}{8EI}\,q_{ij}\right]\right]\begin{Bmatrix}v_2\\ \theta_2\\ v_3\\ \theta_3\end{Bmatrix} = \begin{Bmatrix}0\\0\\0\\0\end{Bmatrix}$$

$$m_{ij} = \begin{bmatrix} 312 & 0 & 54 & -6.5\ell \\ 0 & 2\ell^2 & 6.5\ell & -.75\ell^2 \\ 54 & 6.5\ell & 156 & -11\ell \\ -6.5\ell & -.75\ell^2 & -11\ell & \ell^2 \end{bmatrix}$$

$$k_{ij} = \begin{bmatrix} 24 & 0 & -12 & 3\ell \\ 0 & 2\ell^2 & -3\ell & .5\ell^2 \\ -12 & -3\ell & 12 & -3\ell \\ 3\ell & .5\ell^2 & -3\ell & \ell^2 \end{bmatrix}$$ without elastic foundation

add to k_{ij}, $Q_{ij} = +Q_{ij} =$ above eq.

10-32 Cont

```
»m

m =

  312.0000         0   54.0000   -6.5000
         0    2.0000    6.5000   -0.7500
   54.0000    6.5000  156.0000  -11.0000
   -6.5000   -0.7500  -11.0000    1.0000

»k

k =

   25.8600    0.1310  -11.3600    2.9220
    0.1310    2.0120   -2.9230    0.4910
  -11.3600   -2.9230   13.8600   -3.1310
    2.9220    0.4910   -3.1310    1.0120



a =

    0.0829    0.0052   -0.0822    0.3817
    0.0052    1.0060   -0.5986    1.2847
   -0.0822   -0.5986    0.4697   -1.3314
    0.3817    1.2847   -1.3314    6.4735

»[v,d]=eig(a)

v =

   -0.0544    0.0949    0.6111   -0.7839
   -0.2215   -0.8779    0.3740    0.2007
    0.2118    0.3746    0.6968    0.5739
   -0.9503    0.2827    0.0331    0.1260

d =

    7.0915         0         0         0
         0    0.8472         0         0
         0         0    0.0131         0
         0         0         0    0.0804
```

$\lambda = \dfrac{\omega^2 m \ell^4}{4720\, EI}$

$\omega = 218 \sqrt{\dfrac{EI}{m\ell^4}}$

75.45

23.23

9.378

10-33

Same as Prob 10-32 but with pin at ①

$\therefore v_1 = 0, \quad \theta_1 \neq 0$

$$K = \frac{8EI}{\ell^3}\begin{bmatrix} 1 & -3 & .5 & 0 & 0 \\ -3 & 24 & 0 & -12 & 3 \\ .5 & 0 & 2 & -3 & .5 \\ 0 & -12 & -3 & 12 & -3 \\ 0 & 3 & .5 & -3 & 1 \end{bmatrix} \cdot \begin{Bmatrix} \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix}$$

with $\ell = 1$ inside matrix

$$M = \frac{m\ell}{840}\begin{bmatrix} 1 & 6.5 & -.75 & 0 & 0 \\ 6.5 & 312 & 0 & 54 & -6.5 \\ -.75 & 0 & 2 & 6.5 & -.75 \\ 0 & 54 & 6.5 & 156 & -11 \\ 0 & -6.5 & -.75 & -11 & 1 \end{bmatrix}$$

with $\ell = 1$, inside matrix

$$\begin{Bmatrix} Q_i \end{Bmatrix} = -\left(\frac{k\ell^3}{8EI}\right)\left(\frac{8EI}{\ell^3}\right)\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & .1857 & .0131 & .0643 & -.007736 \\ 0 & .0131 & .001191 & .007738 & -.000893 \\ 0 & .0643 & .007738 & .1857 & -.013075 \\ 0 & -.00738 & -.000893 & -.013075 & .001191 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix}$$

New Stiffness with elastic foundation with $\frac{k\ell^3}{8EI} = 10$

$$\frac{8EI}{\ell^3}\begin{bmatrix} 1 & -3 & .5 & 0 & 0 \\ -3 & 25.86 & .131 & -11.357 & 2.923 \\ .5 & .131 & 2.0119 & -2.923 & -.491 \\ 0 & -11.357 & -2.923 & 13.857 & -3.1307 \\ 0 & 2.923 & .491 & -3.1307 & 1.0119 \end{bmatrix}$$

0-33 CONT.

```
>>m

m =

    1.0000     6.5500    -0.7500          0          0
    6.5000   312.0000          0    54.0000    -6.5000
   -0.7500          0     2.0000     6.5000    -0.7500
         0    54.0000     6.5000   156.0000   -11.0000
         0    -6.5000    -0.7500   -11.0000     1.0000

>>k

k =

    1.0000    -3.0000     0.5000          0          0
   -3.0000    25.8600     0.1310   -11.3600     2.9230
    0.5000     0.1310     2.0120    -2.9230     0.4910
         0   -11.3600    -2.9230    13.8600    -3.1310
         0     2.9230     0.4910    -3.1310     1.0120

a =

    1.0000    -0.5822     1.2269    -0.3532     0.5713
   -0.5822     0.4016    -0.6448     0.0946     0.1294
    1.2269    -0.6448     2.7725    -1.2088     2.3238
   -0.3532     0.0946    -1.2088     0.7127    -1.7975
    0.5713     0.1294     2.3238    -1.7975     7.6275

>>[v,d]=eig(a)

v =

    0.1311    -0.4785     0.6513     0.5105    -0.2625
   -0.0268     0.3343    -0.3957     0.4519    -0.7258
    0.3888    -0.6818    -0.4418    -0.3241    -0.2893
   -0.2458     0.1319     0.4262    -0.6481    -0.5661
    0.8778     0.4206     0.2057    -0.1004    -0.0133

d =

    9.2416          0          0          0          0
         0     2.7500          0          0          0
         0          0     0.4707          0          0
         0          0          0     0.0418          0
         0          0          0          0     0.0101
```

$$\lambda = \frac{\omega^2 m l^4}{6720 EI}$$

$\omega =$ 249 $\sqrt{\frac{EI}{ml^4}}$

135.6

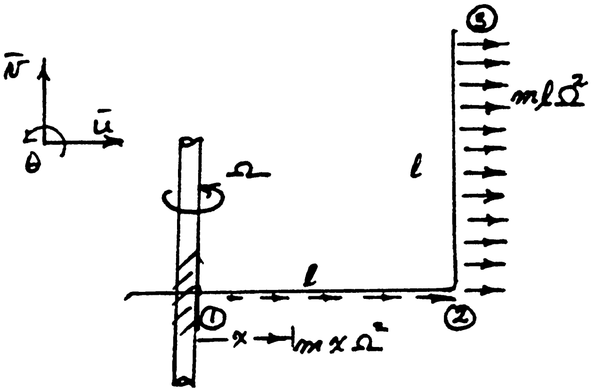
56.21

16.76

8.267

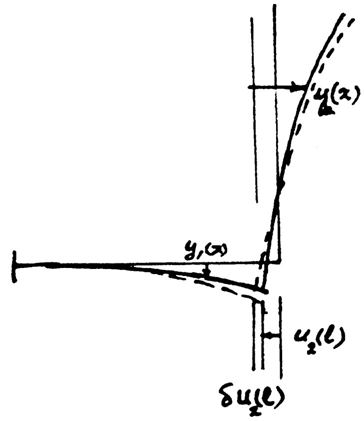
10 - 34

global coordinates



Eq. 10.4-7 $\sim$ K

Eq. 10.4-8 $\sim$ M



Element ①-②

$$\text{Load} = \Omega^2 m x \, dx$$

$$\text{virtual displ} = \int_0^x y_1' \, \delta y_1' \, dr$$

$$= -\sum_i \sum_j q_i \delta q_i \int_0^x \phi_i' \phi_j' \, dr$$

$$\delta W = -\Omega^2 m \sum_i \sum_j q_j \delta q_i \int_0^\ell x \int_0^x \phi_i' \phi_j' \, dr \, dx$$

$$Q_i = -\Omega^2 m \sum_j q_j \int_0^\ell x \int_0^x \phi_i' \phi_j' \, dr \, dx$$

Element ②-③

$$\text{Load} = \Omega^2 \ell_1 m \, dx$$

$$\text{virtual displ.} = \delta y_2(x) - \delta u_1(\ell)$$

$$\delta W = \Omega^2 \ell_1 m \left[\int_0^\ell \delta y_2(x) \, dx - \int_0^\ell y_1' \, \delta y_1' \, dr \right]$$

$$= \Omega^2 \ell_1 m \left[\sum \delta q_j \int_0^\ell \phi_j \, dx - \sum_i \sum_j q_j \, \delta q_i \int_0^\ell \phi_i' \phi_j' \, dx \right]$$

$$Q_i = \frac{\delta W}{\delta q_i} = \Omega^2 \ell_1 m \left[\sum_j \int_0^\ell \phi_j \, dx - \sum_j q_j \int_0^\ell \phi_i' \phi_j' \, dx \right]$$

See Ex 10.8-2 for evaluation of integrals

$$y(x) = \phi_1(x) v_1 + \phi_2(x) \theta_1 + \phi_3(x) v_2 + \phi_4(x) \theta_2 \quad \text{for any element betw sta. 1 and 2.}$$

10-34 cont.

Stiffness without rotation, $\Omega = 0$ — Eq. 10.4-7

$\alpha = 0$
$c = 1$
$s = 0$

$$K_{12} = \frac{EI}{l^3}\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & 0 & 12 & -6 \\ \cdot & \cdot & \cdot & 0 & -6 & 4 \end{bmatrix}\begin{Bmatrix} \cdot \\ \cdot \\ \cdot \\ 0 \\ \bar{v}_2 \\ \theta_2 \end{Bmatrix}$$

$\alpha = 90°$
$c = 0$
$s = 1$

$$K_{23} = \frac{EI}{l^3}\begin{bmatrix} x & x & x & x & x & x \\ x & R & 0 & 0 & -R & 0 \\ x & 0 & 4 & 6 & 0 & 2 \\ x & 0 & 6 & 12 & 0 & 6 \\ x & -R & 0 & 0 & R & 0 \\ x & 0 & 2 & 6 & 0 & 4 \end{bmatrix}\begin{Bmatrix} \bar{u}_2 = 0 \\ \bar{v}_2 \\ \theta_2 \\ \bar{u}_3 \\ v_3 \\ \theta_3 \end{Bmatrix}$$

Assembled — since $\bar{v}_2 = \bar{v}_3$ add columns 2 and 5 and delete row 5

$$K = \frac{EI}{l^3}\begin{vmatrix} x & x & x & x & x & x \\ x & (12+R) & -6 & 0 & -R & 0 \\ x & -6 & 8 & 6 & 0 & 2 \\ x & 0 & 6 & 12 & 0 & 6 \\ x & -R & 0 & 0 & R & 0 \\ x & 0 & 2 & 6 & 0 & 4 \end{vmatrix}\begin{matrix} \\ \bar{v}_2 \\ \theta_2 \\ \bar{u}_3 \\ \bar{v}_3 \\ \theta_3 \end{matrix} = \frac{EI}{l^3}\begin{bmatrix} -12 & -6 & 0 & 0 \\ -6 & 8 & 6 & 2 \\ 0 & 6 & 12 & 6 \\ 0 & 2 & 6 & 4 \end{bmatrix}\begin{Bmatrix} \bar{v}_2 \\ \theta_2 \\ \bar{u}_3 \\ \theta_3 \end{Bmatrix}$$

Mass matrix Eq. 10.4-8

$\alpha = 0$
$c = 1$
$s = 0$

$$M_{12} = \frac{ml}{420}\begin{bmatrix} x & x & x & x & x & x \\ x & 156 & 22 & 0 & 54 & -13 \\ x & 22 & 4 & 0 & 13 & -3 \\ x & 0 & 0 & N & 0 & 0 \\ x & 54 & 13 & 0 & 156 & -22 \\ x & -13 & -3 & 0 & -22 & 4 \end{bmatrix}\begin{matrix} u_1 = 0 \\ v_1 = 0 \\ \theta_1 = 0 \\ u_2 = 0 \\ v_2 \\ \theta_2 \end{matrix} \quad N = 140$$

$\alpha = 90$
$c = 0$
$s = 1$

$$M_{2\text{-}3} = \frac{ml}{420}\begin{bmatrix} x & x & x & x & x & x \\ x & N & 0 & 0 & \tfrac{1}{2}N & 0 \\ x & 0 & 4 & -13 & 0 & -3 \\ x & 0 & -13 & 156 & 0 & 22 \\ x & \tfrac{1}{2}N & 0 & 0 & N & 0 \\ x & 0 & -3 & 22 & 0 & 4 \end{bmatrix}\begin{matrix} \bar{v}_2 \\ \theta_2 \\ \bar{u}_3 \\ \bar{v}_3 \\ \theta_3 \end{matrix}$$

with $\bar{v}_2 = \bar{v}_3$ add columns 2 & 5 & delete row 5

$$M = \frac{ml}{420}\begin{bmatrix} 366 & -22 & 0 & 0 \\ -22 & 8 & -13 & -3 \\ 0 & -13 & 156 & 22 \\ 0 & -3 & 22 & 4 \end{bmatrix}\begin{Bmatrix} \bar{v}_2 \\ \theta_2 \\ \bar{u}_3 \\ \theta_3 \end{Bmatrix}$$

10-34 CONT.

$\Omega = 0$

```
»m=[366 -22 0 0 ; -22 8 -13 -3; 0 -13 156 22; 0 -3 22 4]

m =

   366   -22     0     0
   -22     8   -13    -3
     0   -13   156    22
     0    -3    22     4

»k=[ 12 -6 0 0 ; -6 8 6 2 ; 0 6 12 6 ; 0 2 6 4]

k =

    12    -6     0     0
    -6     8     6     2
     0     6    12     6
     0     2     6     4

a =

    0.0328   -0.1068   -0.0470   -0.0711
   -0.1068    1.0965    0.6856    1.3326
   -0.0470    0.6856    0.4829    1.2102
   -0.0711    1.3326    1.2102    5.4162

»[v,d]=eig(a)

v =

    0.5728    0.8135    0.0993   -0.0176
    0.4220   -0.1883   -0.8417    0.2792
   -0.7002    0.5464   -0.3957    0.2340
    0.0602   -0.0654    0.3537    0.9311

d =

    0.0041         0         0         0
         0    0.0317         0         0
         0         0    0.8714         0
         0         0         0    6.1213
```

$\lambda = \dfrac{\omega^2 m l^4}{420 EI}$

$\omega =$ 1.309
3.647
19.13
50.70

10 – 34 cont

Generalized force Q_i

Element 1-2

$$-m\Omega^2\ell\left[\begin{array}{c|cc} & & \\ - - - - & .42857 & -.064286 \\ & -.064286 & .023810 \end{array}\right]\begin{array}{c} \\ \bar{v}_2 \\ \theta_2 \end{array}$$

Element 2-3

$$m\Omega^2\ell\sum_j\int_0^\ell \varphi_j\,dx = \Omega^2 m\ell\begin{Bmatrix} .50\,\ell \\ .0833\,\ell^2 \\ .50\,\ell \\ -.0833\,\ell^2 \end{Bmatrix}$$ must remain on right side of eq. as force & moment

$$\delta u_1(\ell) = -m\Omega^2\ell\sum_j q_j\int_0^\ell \phi_i'\phi_j'\,dx = -m\Omega^2\ell\left[\begin{array}{cc|cc} 1.2 & .1\ell & -1.2 & .1\ell \\ & .133\,\ell^2 & -.1\ell & -.0333\,\ell^2 \\ \hline & & 1.2 & -.1\ell \\ & & -.1\ell & .1333\,\ell^2 \end{array}\right]\begin{array}{c} 0 \\ 0 \\ \bar{v}_2 \\ \theta_2 \end{array}$$

Total Stiffness

$$\frac{EI}{\ell^3}\begin{bmatrix} 12 & -6 & 0 & 0 \\ -6 & 8 & 6 & 2 \\ 0 & 6 & 12 & 6 \\ 0 & 2 & 6 & 4 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} + m\Omega^2\ell\left[\begin{array}{c|cc} - - - & .42853 & -.06428 \\ & -.06428 & .02381 \end{array}\right]\begin{Bmatrix} \cdots \\ \bar{v}_2 \\ \theta_2 \end{Bmatrix}$$

$$+\, m\Omega^2\ell\left|\begin{array}{c|cc} - - - - & 1.2 & -.1 \\ & -.1 & .1333 \end{array}\right|\begin{Bmatrix} \bar{v}_2 \\ \theta_2 \end{Bmatrix}$$

To combine with $\frac{EI}{\ell^3}$ multiply by $\frac{\ell^3}{EI}$ to obtain $\left(\frac{m\Omega^2\ell^4}{EI}\right)$ chosen as 1.0

$$= \frac{EI}{\ell^3}\left[\begin{array}{cc|cc} 13.628 & -6.16428 & 0 & 0 \\ -6.16428 & 8.15714 & 6 & 2 \\ \hline 0 & 6 & 12 & 6 \\ 0 & 2 & 6 & 4 \end{array}\right]\begin{Bmatrix} v_2 \\ \theta_2 \\ \bar{u}_3 \\ \theta_3 \end{Bmatrix}$$ with

Input into Choljac. with Ω, ie $\frac{m\Omega^2\ell^4}{EI} = 1$, $\lambda = \frac{\omega^2 m\ell^4}{420\,EI}$

10·34 Cont. $\frac{m\Omega^2 \ell^4}{EI} = 1.0$

m matrix as previous case

```
»k=[13.63 -6.164 0 0 ; -6.164 8.157 6 2 ; 0 6 12 6 ; 0 2 6 4]

k =

   13.6300   -6.1640         0         0
   -6.1640    8.1570    6.0000    2.0000
         0    6.0000   12.0000    6.0000
         0    2.0000    6.0000    4.0000

»a=qin'*k*qin

a =

    0.0372   -0.1081   -0.0476   -0.0720
   -0.1081    1.1180    0.6951    1.3469
   -0.0476    0.6951    0.4871    1.2165
   -0.0720    1.3469    1.2165    5.4257

»[v,d]=eig(a)

v =

   -0.0178    0.0992   -0.5175    0.8497
    0.2821   -0.8419   -0.4318   -0.1588
    0.2349   -0.3928    0.7359    0.4989
    0.9300    0.3565   -0.0648   -0.0616
```

$\lambda = \frac{\omega^2 m \ell^4}{420 EI}$

```
d =

    6.1429         0         0         0
         0    0.8846         0         0
         0         0    0.0057         0
         0         0         0    0.0347
```

$\omega =$ 50.8
19.276
3.8187
1.544

10 - 35

From Eq. 10.2-1 and Sec. 10.6 the stiffness for the 1^{st} element of length $\ell/2$ with spring K at the pinned end is

$$\frac{8EI}{\ell^3}\begin{bmatrix} 4\left(\frac{\ell}{2}\right)^2 & -6\left(\frac{\ell}{2}\right) & 2\left(\frac{\ell}{2}\right)^2 \\ -6\left(\frac{\ell}{2}\right) & 12 & -6\left(\frac{\ell}{2}\right) \\ 2\left(\frac{\ell}{2}\right)^2 & -6\left(\frac{\ell}{2}\right) & 4\left(\frac{\ell}{2}\right)^2 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} + K\theta_1$$

$$= \frac{8EI}{\ell^3}\begin{bmatrix} \left(\ell^2 + \frac{K\ell^3}{8EI}\right) & -3\ell & .5\ell^2 \\ -3\ell & 12 & -3\ell \\ .5L^2 & -3\ell & \ell^2 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

When rotated, the generalized force given in Ex. 10.8-2 for element of length ℓ, requires adding the following term

$$m\Omega^2\frac{\ell}{2}\begin{bmatrix} .05714\left(\frac{\ell}{2}\right)^2 & -.01429\left(\frac{\ell}{2}\right) & -.009524\left(\frac{\ell}{2}\right)^2 \\ & .4286 & -.06429\left(\frac{\ell}{2}\right) \\ & & .02381\left(\frac{\ell}{2}\right)^2 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

10 − 36 10 − 37 Discussed in Example 10.8−4 (text)

10 − 38 Let $\ell = 1.0$ inside matrices, $v_1 = \theta_1 = 0$

$$K_{1-2} = \frac{27EI}{\ell^3}\begin{array}{c c} \left[\begin{array}{c|cc} - & - & - \\ & 12 & -6 \\ & -6 & 4 \end{array}\right] & \begin{array}{c} \\ v_2 \\ \theta_2 \end{array} \end{array}$$

$$K_{2-3} = \frac{27EI}{\ell^3}\begin{array}{c c} \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & 6 \\ 6 & 2 & -6 & 4 \end{bmatrix} & \begin{array}{c} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{array} \end{array} \qquad K_{3-4} = \frac{27EI}{\ell^3}\begin{array}{c c} \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} & \begin{array}{c} v_3 \\ \theta_3 \\ v_4 \\ \theta_4 \end{array} \end{array}$$

Asseble

$$K = \frac{27EI}{\ell^3}\begin{bmatrix} 24 & 0 & -12 & 6 & 0 & 0 \\ 0 & 8 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \\ v_4 \\ \theta_4 \end{Bmatrix}$$

Assemble mass

$$M = \frac{m\ell}{420 \times 3}\begin{bmatrix} 312 & 0 & 54 & -13 & 0 & 0 \\ 0 & 8 & 13 & -3 & 0 & 0 \\ 54 & 13 & 312 & 0 & 54 & -13 \\ -13 & -3 & 0 & 8 & 13 & -3 \\ 0 & 0 & 54 & 13 & 156 & -22 \\ 0 & 0 & -13 & -3 & -22 & 4 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \\ v_4 \\ \theta_4 \end{Bmatrix}$$

The above 6×6 matrices are inputted into Choljac with $\lambda = \dfrac{\omega^2 m \ell^4}{420 \times 3 \times 27 EI} = \dfrac{\omega^2 m \ell^4}{34020\, EI}$

To obtain reduced stiffness, see Ex 10.5−2 (text)

1st rearrange rows with displacements in first three rows and angles in rows 4, 5, 6.

Then rearrange columns to conform to new column vector

10-58 Cont.

For stiffness matrix

Rearranged rows

$$\left[\begin{array}{cccccc} 24 & 0 & -12 & 6 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ \hline 0 & 8 & -6 & 2 & 0 & 0 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{array}\right] \begin{array}{c} v_2 \\ v_3 \\ v_4 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{array}$$

Rearrange columns

$$\left[\begin{array}{ccc|ccc} 24 & -12 & 0 & 0 & 6 & 0 \\ -12 & 24 & -12 & -6 & 0 & 6 \\ 0 & -12 & 12 & 0 & -6 & -6 \\ \hline 0 & -6 & 0 & 8 & 2 & 0 \\ 6 & 0 & -6 & 2 & 8 & 2 \\ 0 & 6 & -6 & 0 & 2 & 4 \end{array}\right] \begin{Bmatrix} v_2 \\ v_3 \\ v_4 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix}$$

$$= \left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array}\right] \begin{Bmatrix} V \\ \hline \theta \end{Bmatrix}$$

For the mass Matrix

The rearranged mass matrix is

$$\left[\begin{array}{ccc|ccc} 312 & 54 & 0 & 0 & -13 & 0 \\ 54 & 312 & 54 & 13 & 0 & -13 \\ 0 & 54 & 156 & 0 & 13 & 22 \\ \hline 0 & 13 & 0 & 8 & -3 & 0 \\ -13 & 0 & 13 & -3 & 8 & -3 \\ 0 & -13 & -22 & 0 & -3 & 4 \end{array}\right] \begin{Bmatrix} v_2 \\ v_3 \\ v_4 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix} = \left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array}\right] \begin{Bmatrix} V \\ \hline \theta \end{Bmatrix}$$

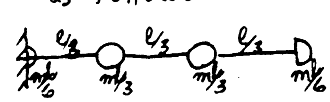
Assume $M_{12} = M_{21} = M_{22} = 0$

$$\left[\begin{array}{c|c} M_{11} & 0 \\ \hline 0 & 0 \end{array}\right] \begin{Bmatrix} \ddot{V} \\ \ddot{\theta} \end{Bmatrix} + \left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array}\right] \begin{Bmatrix} V \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$M_{11} \ddot{V} + K^{*} V = 0 \quad = 3\times3 \text{ equation}$$

where $K^{*} = K_{11} - K_{12} K_{22}^{-1} K_{21}$ = reduced stiffness

Although $M_{11} = \dfrac{m\ell}{420\times3}\begin{bmatrix} 312 & 54 & 0 \\ 54 & 312 & 54 \\ 0 & 54 & 156 \end{bmatrix}$ we can choose discrete masses as follows



10 – 38 cont.

ie. assume half the mass of each element at the ends

$$M_{11} = m\ell \begin{bmatrix} .333 & 0 & 0 \\ 0 & .333 & 0 \\ 0 & 0 & .1665 \end{bmatrix}$$

For K^*

$$K_{22}^{-1} = \begin{bmatrix} 8 & 2 & 0 \\ 2 & 8 & 2 \\ 0 & 2 & 4 \end{bmatrix}^{-1} = \frac{1}{208}\begin{bmatrix} 28 & -8 & 4 \\ -8 & 32 & -16 \\ 4 & -16 & 60 \end{bmatrix}$$

$$K_{12}K_{22}^{-1}K_{21} = \begin{bmatrix} 0 & 6 & 0 \\ -6 & 0 & 6 \\ 0 & -6 & -6 \end{bmatrix}\begin{bmatrix} 28 & -8 & 4 \\ -8 & 32 & -16 \\ 4 & -16 & 60 \end{bmatrix}\begin{bmatrix} 0 & -6 & 0 \\ 6 & 0 & -6 \\ 0 & 6 & -6 \end{bmatrix}\frac{1}{208}$$

$$= \begin{bmatrix} 1152 & -288 & -576 \\ -288 & 2880 & -1728 \\ -576 & -1728 & 2160 \end{bmatrix}\frac{1}{208} = \begin{bmatrix} 5.538 & -1.3846 & -2.769 \\ -1.3846 & 13.846 & -8.308 \\ -2.769 & -8.308 & 10.385 \end{bmatrix}$$

$$K^* = \begin{bmatrix} 24 & -12 & 0 \\ -12 & 24 & -12 \\ 0 & -12 & 12 \end{bmatrix} - \begin{bmatrix} 5.538 & -1.3846 & -2.769 \\ -1.3846 & 13.846 & -8.308 \\ -2.769 & -8.308 & 10.385 \end{bmatrix}$$

$$= \begin{bmatrix} 18.462 & -10.615 & 2.769 \\ -10.615 & 10.154 & -3.692 \\ 2.769 & -3.692 & 1.615 \end{bmatrix}\frac{27EI}{\ell^3}$$

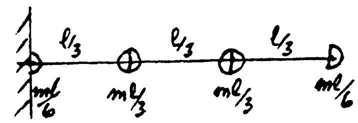
$$\lambda = (\omega^2 m\ell)\frac{\ell^3}{27EI} = \frac{\omega^2}{27}\frac{m\ell^4}{EI}$$

$$\omega = \sqrt{27\lambda}\sqrt{\frac{EI}{m\ell^4}}$$

10-38 cont

reduced stiffness with discrete masses

```
»m=[.333 0 0 ; 0 .333 0 ; 0 0 .1665]

m =

    0.3330         0         0
         0    0.3330         0
         0         0    0.1665

»k=[ 18.46 -10.61 2.769; -10.61 10.15 -3.692; 2.769 -3.692 1.615]

k =

   18.4600  -10.6100    2.7690
  -10.6100   10.1500   -3.6920
    2.7690   -3.6920    1.6150
```




```
a =

   55.4354  -31.8619   11.7596
  -31.8619   30.4805  -15.6795
   11.7596  -15.6795    9.6997

»[v,d]=eig(a)

v =

    0.7873   -0.5901    0.1786
   -0.5634   -0.5710    0.5971
    0.2503    0.5708    0.7820

d =

   81.9738         0         0
         0   13.2281         0
         0         0    0.4137
```

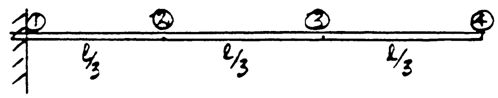
$$\lambda = \frac{\omega^2 m l^4}{27EI}$$

$$\omega = \begin{matrix} 47.05 \\ 18.89 \\ 3.347 \end{matrix} \sqrt{\frac{EI}{ml^4}}$$

10-38 cont.

m =

312	0	54	-13	0	0
0	8	13	-3	0	0
54	13	312	0	54	-13
-13	-3	0	8	13	-3
0	0	54	13	156	-22
0	0	-13	-3	-22	4



k =

24	0	-12	6	0	0
0	8	-6	2	0	0
-12	-6	24	0	-12	6
6	2	0	8	-6	2
0	0	-12	-6	12	-6
0	0	6	2	-6	4

a =

0.0769	0	-0.0545	0.1691	-0.0555	0.0912
0	1.0000	-0.4004	0.7812	-0.2136	0.3415
-0.0545	-0.4004	0.2469	-0.3327	0.0045	0.2650
0.1691	0.7812	-0.3327	1.9215	-0.9207	1.8330
-0.0555	-0.2136	0.0045	-0.9207	0.5897	-1.5889
0.0912	0.3415	0.2650	1.8330	-1.5889	7.1247

»[v,d]=eig(a)

v =

0.0183	-0.0567	-0.1449	-0.6481	0.7052	-0.2409
0.0837	-0.5490	0.7351	0.2340	0.2639	-0.1636
0.0130	0.2971	-0.2094	0.5674	0.2667	-0.6889
0.3126	-0.6894	-0.4922	-0.0446	-0.3072	-0.2974
-0.2321	0.1647	0.3399	-0.4345	-0.5128	-0.5930
0.9170	0.3237	0.1926	-0.1113	-0.0671	-0.0192

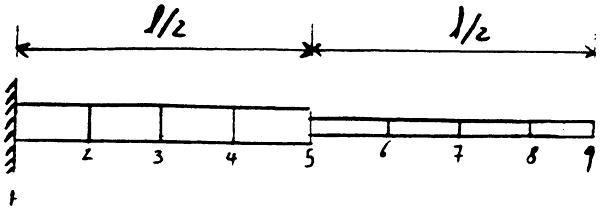
$$\lambda = \frac{\omega^2 m \ell^4}{34020 EI}$$

d =

8.1884	0	0	0	0	0
0	2.0602	0	0	0	0
0	0	0.5817	0	0	0
0	0	0	0.1147	0	0
0	0	0	0	0.0144	0
0	0	0	0	0	0.0004

$\omega = 527.6$, 264.5, 140.796, 63.520, 22.11, 3.516 $\sqrt{\frac{EI}{m\ell^4}}$

10 – 40



$l = 1$.

The mass and stiffness matrices for the elements in the first (larger in diameter) section are given as

$$\frac{1}{12}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad 16\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

For the other section, they are given as

$$\frac{1}{24}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad 8\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$u_1 = 0$. Therefore, the mass and stiffness for the system are given as (neglect u_1)

$$M = \frac{1}{24}\begin{bmatrix} 8 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 8 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 8 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 6 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}, \quad K = 8\begin{bmatrix} 4 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 4 & -2 & 0 & 0 & 0 & 0 & 0 \\ & -2 & 4 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 3 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Let $u = [u_2 \; u_3 \; u_4 \; u_5 \; u_6 \; u_7 \; u_8 \; u_9]^T$. Therefore, the equation for the longitudinal vibration is

$$M\ddot{u} + Ku = 0.$$

The natural frequencies are the square roots of the generalized eigenvalues of M and K, i.e., $K - \omega^2 M = 0$. Solving for ω, we get

$$\omega = 19.186,\ 17.628,\ 14.040,\ 11.603,\ 8.096,\ 6.049,\ 1.354,\ 3.130$$

You can get the same answers from the program pin_bar.m
enter $m = [2, 2, 2, 2, 1, 1, 1, 1]$
$A = [2, 2, 2, 2, 1, 1, 1, 1]$

For the two values to agree recall that the above values have ~~are~~ been multiplied by $\frac{1}{\sqrt{2}}$

10-41 cont output from pin_beam.m

```
omega =

   54.5345         0         0         0
         0   18.7893         0         0
         0         0    0.8794         0
         0         0         0    5.5554
```

2-element beam

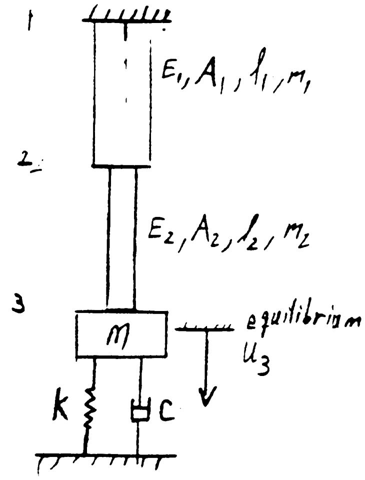
```
omega =

  Columns 1 through 7

  538.6989         0         0         0         0         0         0
         0  390.6837         0         0         0         0         0
         0         0  219.6134         0         0         0         0
         0         0         0  297.0578         0         0         0
         0         0         0         0  160.7123         0         0
         0         0         0         0         0  117.0056         0
         0         0         0         0         0         0   75.8829
         0         0         0         0         0         0         0
         0         0         0         0         0         0         0
         0         0         0         0         0         0         0
         0         0         0         0         0         0         0
         0         0         0         0         0         0         0

  Columns 8 through 12

         0         0         0         0         0
         0         0         0         0         0
         0         0         0         0         0
         0         0         0         0         0
         0         0         0         0         0
         0         0         0         0         0
         0         0         0         0         0
   50.7158         0         0         0         0
         0   30.4202         0         0         0
         0         0   15.4525         0         0
         0         0         0    5.5100         0
         0         0         0         0    0.8790
```

6-element beam

see example 10-5.1

10-42

The mass and stiffness matrices of the system are given as (Ignore U_1 since it is equal to zero) let $u = [U_2 \quad U_3]$



$$M = \begin{bmatrix} \frac{m_1 l_1}{3} + \frac{m_2 l_2}{3} & \frac{m_2 l_2}{6} \\ \frac{m_2 l_2}{6} & \frac{m_2 l_2}{3} + m \end{bmatrix}$$

$$K = \begin{bmatrix} \frac{E_1 A_1}{l_1} + \frac{E_2 A_2}{l_2} & -\frac{E_2 A_2}{l_2} \\ -\frac{E_2 A_2}{l_2} & \frac{E_2 A_2}{l_2} + k \end{bmatrix}$$

There is also a damping force at 3 equal to $c\dot{U}_3$ opposite to the direction of motion. Therefore, the equation of motion is written as:

$$\frac{1}{3}\begin{bmatrix} m_1 l_1 + m_2 l_2 & m_2 l_2/2 \\ m_2 l_2/2 & m_2 l_2 + 3m \end{bmatrix}\begin{bmatrix} \ddot{U}_2 \\ \ddot{U}_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}\begin{bmatrix} \dot{U}_2 \\ \dot{U}_3 \end{bmatrix} + \begin{bmatrix} \frac{E_1 A_1}{l_1} + \frac{E_2 A_2}{l_2} & -\frac{E_2 A_2}{l_2} \\ -\frac{E_2 A_2}{l_2} & \frac{E_2 A_2}{l_2} + k \end{bmatrix}\begin{bmatrix} U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

10-43

For a single beam $v_2 = \theta_2 = M_1 = 0$, we have

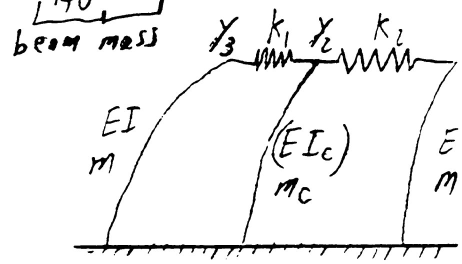
$$\begin{bmatrix} F_1 \\ M_1 \end{bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l \\ 6l & 4l^2 \end{bmatrix}\begin{bmatrix} v_1 \\ \theta_1 \end{bmatrix}$$

$$\therefore \quad M_1 = 0 \Rightarrow 6l v_1 + 4l^2\theta_1 = 0 \Rightarrow \theta_1 = -\frac{3}{2l} v_1$$

$$\therefore \quad F_1 = \frac{EI}{l^3}\left[12 v_1 + 6l\theta_1\right] = \underbrace{\frac{3EI}{l^3}}_{\text{beam stiffness}} v_1 \ (\text{cantilever})$$

The mass is found by multiplying the first row of the mass matrix by $[\ddot{v}_1 \ \ddot{\theta}_1 \ \ddot{v}_2 \ \ddot{\theta}_2]^T$

$$\Rightarrow \frac{ml}{420}\left[156\ddot{v}_1 + 22l\ddot{\theta}_1\right] = \underbrace{\frac{41}{140} ml}_{\text{beam mass}} \ddot{v}_1$$



Equations of motion

1) $\frac{41}{140} ml\ddot{y}_1 = -\frac{3EI}{l^3} y_1 - k_2(y_1 - y_2)$

$$\Rightarrow \frac{41}{140} ml\ddot{y}_1 + \left(k_2 + \frac{3EI}{l^3}\right) y_1 = k_2 y_2$$

2) $\frac{41}{140} m_c l\ddot{y}_2 = k_2(y_1 - y_2) - k_1(y_2 - y_3) - \frac{3(EI)_c}{l^3} y_2$

$$\Rightarrow \frac{41}{140} m_c l\ddot{y}_2 + \left(k_1 + k_2 + \frac{3(EI)_c}{l^3}\right) y_2 = k_1 y_3 + k_2 y_1$$

3) $\frac{41}{140} ml\ddot{y}_3 + \left(k_1 + \frac{3EI}{l^3}\right) y_3 = k_1 y_2$

If $y_1 = y_2 = y_3$ the forces from k_1 and k_2 are zero.

$\therefore$ Free vibration for the three beams,

$\Rightarrow$ They must vibrate with the same frequency ω

$$\omega = \sqrt{\frac{3EI/l^3}{\frac{41}{140} ml}} = \sqrt{\frac{420EI}{41ml^4}} = \sqrt{\frac{420(EI)_c}{41m_c l^4}}$$

condition. $\therefore$ Impossible if $\frac{(EI)_c}{m_c} \neq \frac{EI}{m}$

11-1

$$m\ddot{x} + kx = P_0$$

$$x = \frac{P_0}{k}(1 - \cos\omega t), \qquad \omega = \sqrt{\frac{k}{m}}$$

$$\therefore\ x_{max} = 2\frac{P_0}{k}$$

11-2

$$\ddot{q}_i + 2\zeta_i\omega_i\dot{q}_i + \omega_i^2 q_i = \frac{P_0}{m}$$

$$q_i = \frac{P_0}{m}\left[1 - \frac{e^{-\zeta_i\omega_i t}}{\sqrt{1-\zeta_i^2}}\cos(\sqrt{1-\zeta_i}\ \omega_i t - \psi_i)\right]$$

$$\tan\psi_i = \frac{\zeta_i}{\sqrt{1-\zeta_i^2}}$$

For small damping $\psi_i \cong 0$

$$\sqrt{1-\zeta_i^2} \cong 1$$

$$\therefore\ q_i \cong \frac{P_0}{k}\left[1 - e^{-\zeta_i\omega_i t}\cos\omega_i t\right] = \frac{P_0}{k}D_i(t)$$

11-3

$$\ddot{q}_i + 2\zeta_i\omega_i\dot{q}_i + \omega_i^2 q_i = \frac{f(t)}{M}\int_0^l \frac{P_0}{l}\phi_i(x)\,dx$$

where $f(x,t) = \frac{P_0}{l}f(t)$ and P_0 = total force

Mode participation factor $= \frac{1}{l}\int_0^l \phi_i(x)\,dx$

11-4 Generalized force $= \frac{P_0}{M_i}\left\{\frac{1}{l}\int_0^l p(x)\phi_i(x)\,dx\right\}f(t)$

(See Sec. 11.1)

$$K_i = \frac{1}{l}\int_0^l p(x)\phi_i(x)\,dx, \qquad p(x) = l\,\delta(x-a)$$

$$\therefore\ K_i = \frac{1}{l}\int_0^l l\,\delta(x-a)\phi_i(x)\,dx = \phi_i(a)$$

$$q_i = \frac{P_0K_i}{M_i\omega_i^2}D_i(t), \qquad y(x,t) = \sum_i q_i\phi_i = \sum_i \frac{P_0\phi_i(a)\phi_i(x)}{M_i\omega_i^2}D_i(t)$$

$$\omega_i^2 = (\beta_i l)^2\frac{EI}{M_i l^3} \qquad \therefore\ y(x,t) = \frac{P_0 l^3}{EI}\sum_i\frac{\phi_i(a)\phi_i(x)}{(\beta_i l)^4}D_i(t)$$

11-5

$$K_i = \lim_{\epsilon\to 0} \frac{1}{l}\int_0^l l^2\left[\frac{\delta(x-a-\epsilon)-\delta(x-a)}{\epsilon}\right]\phi_i(x)\,dx$$

$$= \lim_{\epsilon\to 0} l\left[\frac{\phi_i(a+\epsilon)-\phi_i(a)}{\epsilon}\right] = l\left.\frac{d\phi_i(x)}{dx}\right|_{x=a}$$

$$= \beta_i l\cdot\frac{1}{\beta_i}\left.\frac{d\phi_i(x)}{dx}\right|_{x=a} = \beta_i l\,\phi_i'(a)$$

See Append. D for ϕ'

11-6

$$K_i = \frac{1}{l}\int_0^l l\,\delta(x-\tfrac{l}{2})\phi_i(x)\,dx = \phi_i(\tfrac{l}{2})$$

$\phi_m(x) = \sqrt{2}\sin m\pi\frac{x}{l}$ when normalized to total mass M

$$\phi_m(\tfrac{l}{2}) = \sqrt{2}\sin\frac{m\pi}{2}, \qquad m = 1, 3, 5, \cdots$$

$$\omega_m^2 = (\beta_m l)^4\frac{EI}{Ml^3}, \qquad \beta_m = \frac{m\pi}{l}, \qquad m = 1, 3, 5, \cdots$$

$$y(x,t) = \frac{2P_0 l^3}{EI}\sum_m\left\{\frac{\sin m\frac{\pi}{2}\cdot\sin m\pi\frac{x}{l}}{(m\pi)^4}D_m(t)\right\} \qquad m = 1, 3, 5\cdots$$

11-7 From Prob. 11-5 $\quad K_m = \beta l\,\phi_m'(a), \quad \beta_m = \frac{m\pi}{l}$

$$\phi_m(x) = \sqrt{2}\sin m\pi\frac{x}{l} \qquad \frac{d\phi_m}{dx} = \sqrt{2}\frac{m\pi}{l}\cos\frac{m\pi x}{l}$$

$$\phi_m'(\tfrac{l}{2}) = \frac{1}{\beta_m}\frac{d\phi_m(\frac{l}{2})}{dx} = \frac{\sqrt{2}}{l}\cos\frac{m\pi}{2}, \qquad m = 2, 4, 6, \cdots$$

$$\ddot{q}_m + \omega_m^2 q_m = \frac{m_o}{M}\left[\beta_m l\,\phi_m'(a)\right]f(t)$$

$$q_m = \frac{m_o}{M\omega_m^2}\beta_m l\,\phi_m'(\tfrac{l}{2})D_m(t)$$

$$y(x,t) = \frac{m_o}{M}\sum_m\frac{\beta_m l}{\omega_m^2}\phi_m'(\tfrac{l}{2})\phi_m(x)D_m(t) = \frac{m_o l^3}{EI}\sum_m\frac{\phi_m'(\frac{l}{2})\phi_m(x)}{(\beta_m l)^3}D_m(t)$$

$$= \frac{2m_o l^2}{EI}\sum_{m=2,4,\cdots}\frac{\cos\frac{m\pi}{2}\sin\frac{m\pi x}{l}}{(m\pi)^3}D_m(t)$$

11-8 Only even modes, $n = 2, 6, \cdots$ will give $K_n \neq 0$

$$\phi_n = \sqrt{2}\,\sin\frac{n\pi x}{l}$$

$$K_n = \frac{1}{l}\int_0^{l/2}\sqrt{2}\,\sin\frac{n\pi x}{l}\,dx - \frac{1}{l}\int_{l/2}^{l}\sqrt{2}\,\sin\frac{n\pi x}{l}\,dx = \frac{2\sqrt{2}}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)$$

$$D_n = \omega_n\int_0^t \sin\omega_n(t-\xi)\,d\xi = 1-\cos\omega_n t$$

$$y(x,t) = \sum_n \frac{p_0 l K_n \phi_n}{\omega_n^2 M} D_n = \sum_{n=2,6,\cdots} \frac{p_0 l}{\omega_n^2 M}\,\frac{2\sqrt{2}}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)\sqrt{2}\,\sin\frac{n\pi x}{l}\left(1-\cos\omega_n t\right)$$

$$= \frac{2p_0 l}{\pi M}\sum_{n=2,6,\cdots} 2\left(1-\cos\frac{n\pi}{2}\right)\sin\frac{n\pi x}{l}\left(1-\cos\omega_n t\right)$$

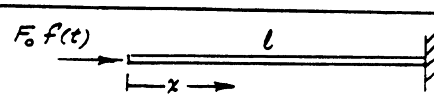
1st mode, $n = 2$

$$y_1(x,t) = \frac{4p_0 l}{\pi M\omega_2^2}\sin\frac{2\pi x}{l}\left(1-\cos\omega_2 t\right)$$

2nd mode, $n = 6$

$$y_2(x,t) = \frac{4p_0 l}{\pi M\omega_6^2}\sin\frac{6\pi x}{l}\left(1-\cos\omega_6 t\right)$$

11-9



Normal mode $u(x) = \mathcal{A}\sin\frac{\omega x}{c} + \mathcal{B}\cos\frac{\omega x}{c}$

$$\frac{du}{dx} = \frac{\omega}{c}\left\{\mathcal{A}\cos\frac{\omega x}{c} - \mathcal{B}\sin\frac{\omega x}{c}\right\}$$

Boundary Cond.

$x = 0$, $\frac{du}{dx} = 0 \quad \therefore \mathcal{A} = 0$

$x = l$, $u = 0 \quad \therefore \mathcal{B}\cos\frac{\omega l}{c} = 0$

$\frac{\omega l}{c} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \cdots$

$$u(x) = \mathcal{B}\cos\frac{\omega x}{c}$$

$$\int_0^l \frac{M}{l}\mathcal{B}^2\cos^2\frac{\omega x}{c}\,dx = M = \frac{\beta^2}{l}\int_0^l \frac{1}{2}\left(1+\cos\frac{2\omega x}{c}\right)dx = \frac{\beta^2}{2}$$

$$\therefore \beta = \sqrt{2} \qquad \phi_n(x) = \sqrt{2}\cos\frac{n\pi}{2}\frac{x}{l}, \qquad n = 1, 3, 5, \cdots$$

11-9 Cont.:

$$\ddot{q}_n + \omega_n^2 q_n = \frac{f(t)}{M}\int_0^\ell F_0\,\delta(x)\sqrt{2}\cos\frac{n\pi}{2}\frac{x}{\ell}\,dx = \frac{F_0\sqrt{2}}{M}f(t)$$

$$q_n = \frac{F_0 K_n}{\omega_n^2 M}D_n(t) = \frac{\sqrt{2}\,F_0}{\left(\frac{n\pi}{2}\frac{c}{\ell}\right)^2 M}D_n(t) = \frac{\sqrt{2}\,F_0 \cancel{M}\ell^{\cancel{2}} D_n(t)}{\left(\frac{n\pi}{2}\right)^2 \cancel{M} E\cancel{\ell} A}$$

$$u(x,t) = \sum_n \phi_n(x)q_n(t) = \frac{2F_0\ell}{EA}\left\{\frac{\cos\frac{\pi x}{2\ell}\cdot D_1(t)}{\left(\frac{\pi}{2}\right)^2} + \frac{\cos\frac{3\pi x}{2\ell}\cdot D_3(t)}{\left(\frac{3\pi}{2}\right)^2} + \cdots\right\}$$

11-10

$$K_n = \frac{1}{\ell}\int_0^\ell \ell\,\delta\left(x-\frac{\ell}{3}\right)\sqrt{2}\cos\frac{n\pi}{2}\frac{x}{\ell}\,dx = \sqrt{2}\cos\frac{n\pi}{6}$$

$n = 1, 3, 5, \cdots$

n				
1	$\cos\frac{\pi}{6}$	$= \cos 30°$	$= .866$	
3	$\cos\frac{3\pi}{6}$	$= \cos 90°$	$= 0$	∴ mode absent
5	$\cos\frac{5\pi}{6}$	$= \cos 150°$	$= -.866$	
7	$\cos\frac{7\pi}{6}$	$= \cos 210°$	$= -.866$	
9	$\cos\frac{9\pi}{6}$	$= \cos 270°$	$= 0$	∴ mode absent
11	$\cos\frac{11\pi}{6}$	$= \cos 330°$	$= .866$	

∴ modes present are 1, 5, 7, 11, 13, ···

11-11

$$K_n = \sqrt{2}\cos\frac{n\pi}{6}$$

$$q_n = \frac{F_0 K_n}{\omega_n^2 M}D_n(t) \qquad \omega_n^2 = \left(\frac{n\pi}{2}\frac{c}{\ell}\right)^2 = \left(\frac{n\pi}{2}\right)^2\frac{EA}{M\ell}$$

$$u(x,t) = \frac{2F_0\ell}{AE}\left\{\frac{0.866}{\left(\frac{\pi}{2}\right)^2}\cos\frac{\pi x}{2\ell}\cdot D_1(t) - \frac{0.866}{\left(\frac{5\pi}{2}\right)^2}\cos\frac{5\pi x}{2\ell}\cdot D_5(t)\right.$$

$$\left. - \frac{0.866}{\left(\frac{7\pi}{2}\right)^2}\cos\frac{7\pi x}{2\ell}\cdot D_7(t) + \frac{0.866}{\left(\frac{11\pi}{2}\right)^2}\cos\frac{11\pi x}{2\ell}\cdot D_{11}(t) + \cdots\right\}$$

11-12 $\quad y(x,t) = q_1 \sin\frac{\pi x}{\ell} + q_2$

$$T = \frac{1}{2}\int_0^\ell m\dot{y}^2 dm = \frac{1}{2}\int_0^\ell (\varphi_1\dot{q}_1 + \varphi_2\dot{q}_2)^2 dx$$

$$= \frac{1}{2} m\ell\left[\frac{1}{2}\dot{q}_1^2 + \frac{4}{\pi}\dot{q}_1\dot{q}_2 + \dot{q}_2^2\right]$$

$$U = \frac{1}{2}\left(\frac{k}{2}\right)y^2(0) + \frac{1}{2}\left(\frac{k}{2}\right)y^2(\ell) + \frac{1}{2}\int_0^\ell EI\, y''^2 dx$$

$$= \frac{1}{2}kq_2^2 + \frac{1}{2}EI\, q_1^2\int_0^\ell \left(\frac{\pi}{\ell}\right)^4 \sin^2\frac{\pi x}{\ell}\, dx$$

$$= \frac{1}{2}kq_2^2 + \frac{1}{2}EI\, q_1^2\left(\frac{\pi}{\ell}\right)^4\frac{\ell}{2} \qquad \text{sub. into } \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial U}{\partial q_i} = 0$$

$$\ddot{q}_1 + \frac{4}{\pi}\ddot{q}_2 + \pi^4\frac{EI}{m\ell^4}q_1 = 0, \quad \text{and} \quad \frac{2}{\pi}\ddot{q}_1 + \ddot{q}_2 + \frac{k}{m\ell}q_2 = 0$$

rewrite

$$\ddot{q}_1 + \frac{4}{\pi}\ddot{q}_2 + \omega_{11}^2 q_1 = 0 \quad \text{and} \quad \frac{2}{\pi}\ddot{q}_1 + \ddot{q}_2 + \omega_{22}^2 q_2 = 0$$

$$\left.\begin{aligned}(\omega_{11}^2 - \omega^2)q_1 - \frac{4}{\pi}\omega^2 q_2 &= 0\\ -\frac{2}{\pi}\omega^2 q_1 + (\omega_{22}^2 - \omega^2)q_2 &= 0\end{aligned}\right\} \quad \omega^4 - \left(\frac{\pi^2}{\pi^2-8}\right)(\omega_{11}^2 + \omega_{22}^2)\omega^2 + \frac{\pi^2\omega_{11}^2\omega_{22}^2}{\pi^2-8} = 0$$

$$\omega^2 = \frac{\omega_{22}^2}{2}\left(\frac{\pi^2}{\pi^2-8}\right)\left[(1+R) \pm \sqrt{(1-R)^2 + \frac{32}{\pi^2}R}\right] \quad \text{where } R = \left(\frac{\omega_{11}}{\omega_{22}}\right)^2$$

Assume $y = \left(b + \sin\frac{\pi x}{\ell}\right)q$

$$T = \frac{1}{2}m\dot{q}^2\int_0^\ell \left(b^2 + 2b\sin\frac{\pi x}{\ell} + \sin^2\frac{\pi x}{\ell}\right)dx = \frac{1}{2}m\dot{q}^2\left(b^2\ell + 4b\frac{\ell}{\pi} + \frac{\ell}{2}\right)$$

$$U = \left[\frac{1}{2}kb^2 + \frac{1}{2}EI\left(\frac{\pi}{\ell}\right)^4\frac{\ell}{2}\right]q^2 \qquad T = U \text{ gives}$$

$$\omega^2 = \frac{\frac{k}{M}b^2 + \frac{1}{2}\omega_{11}^2}{b^2 + \frac{4b}{\pi} + \frac{1}{2}}$$

Momentum = 0 gives

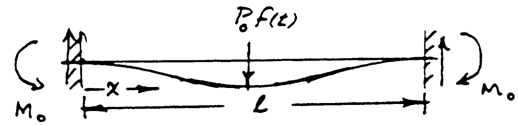
$$kb - m\omega^2\int_0^\ell\left(\sin\frac{\pi x}{\ell} + b\right)dx = 0 \quad \therefore\ kb - m\omega^2\left(-2\frac{\ell}{\pi} + b\ell\right) = 0$$

$$b = \frac{2}{\pi}\left(\frac{\omega^2}{\omega_{22}^2 - \omega^2}\right) = q_2/q_1 \quad \text{or} \quad \omega^2 = \frac{\omega_{22}^2 b}{b + \frac{2}{\pi}}$$

Equating ω^2 $\quad \dfrac{\omega_{22}^2 b}{b + \frac{2}{\pi}} = \dfrac{\omega_{22}^2 b^2 + \frac{1}{2}\omega_{11}^2}{b^2 + \frac{4b}{\pi} + \frac{1}{2}} \quad \therefore\ b^2 + \frac{\pi}{4}(1-R)b + \frac{1}{2}R = 0$

$$b = \frac{\pi}{8}\left[(R-1) \pm \sqrt{(R-1)^2 + \frac{32}{\pi^2}R}\right] \qquad R = \left(\frac{\omega_{11}}{\omega_{22}}\right)^2$$

11-13 $\quad y = \sum_n \phi_n q_n$



Generalized force

$$= P_o f(t) \int_0^l \phi_m(x)\delta(x - \tfrac{l}{2})\,dx = P_o \phi_m(\tfrac{l}{2}) f(t)$$

$$y(\tfrac{l}{2}, t) = \sum_m \phi_m(\tfrac{l}{2}) q_m(t) \qquad \text{where } q_m = \text{solution of}$$

$$\ddot{q}_m + \omega_m^2 q_m = \frac{P_o}{M}\phi_m(\tfrac{l}{2}) f(t)$$

Solution

$$q_m(t) = q_m(0)\cos\omega_m t + \frac{1}{\omega_m}\dot{q}_m(0)\sin\omega_m t + \frac{P_o \phi_m(\frac{l}{2})}{M\omega_m^2}\cdot\omega_m \int_0^t f(\xi)\sin\omega_m(t-\xi)\,d\xi$$

From Appendix D

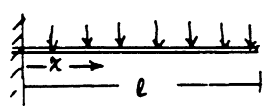
$\phi_1(\frac{l}{2}) = 1.583$, $\qquad \beta_1 l = 4.73 \qquad \phi_1''(0) = 2.0$

$\phi_2(\frac{l}{2}) = 0 \qquad \beta_2 l = 7.85 \qquad \phi_2''(0) = 2.0$

$\phi_3(\frac{l}{2}) = -1.372 \qquad \beta_3 l = 10.99 \qquad \phi_3''(0) = 2.0$

$$M_o = EI\left(\frac{d^2y}{dx^2}\right)_0 = EI\sum_m q_m(t)\left(\frac{d^2\phi_m}{dx^2}\right)_{x=0} = EI\sum_m q_m(t)\beta_m^2\phi_m''(0)$$

11-14



$$K_i = \frac{1}{l}\int_0^l \phi_i(x)\,dx$$

The above integral for K_i can be evaluated from tables in App. D. Add column of $\phi_i(x)$ and mult. by interval $\frac{dx}{l} = 0.04$. 1st mode column sum is 20.57 $\therefore$ $20.57 \times 0.04 = 0.822$. Exact value from integrating Eq. 9.4-12 with proper boundary conditions is 0.783. i.e. use $y(x) = (\cosh\beta x - \cos\beta x) - \left(\frac{\sinh\beta l - \sin\beta l}{\cosh\beta l + \cos\beta l}\right)(\sinh\beta x - \sin\beta x)$

<u>Note:</u> The number 20.57 is too large since it was computed with the largest deflection of $2.0 \times .04 + 1.89 \times .04 + \cdots$ etc. If the sum is started with $1.89 \times .04 + \cdots$ etc, then a figure of 18.57 is obtained. Averaging $\frac{1}{2}(18.57 + 20.57) \times .04$ we obtain the exact value 0.783.

Letting $\alpha_m = \frac{\sinh\beta l - \sin\beta l}{\cosh\beta l + \cos\beta l}$, the integral can be shown to be

$$\int_0^l \phi_m(x)\frac{dx}{l} = \frac{2\alpha_m}{\beta_m l}$$

11-15

From Eq. 11.2-8 $\quad \cancel{\bar{q}}\, M(\omega_1^2-\omega^2) = -k\,\phi_1^2(\tfrac{\ell}{3})\,\cancel{\bar{q}}$

$$1-\left(\frac{\omega}{\omega_1}\right)^2 = -\frac{k}{M}\,\phi_1^2(\tfrac{\ell}{3})\frac{1}{\omega_1^2} \qquad \left(\frac{\omega}{\omega_1}\right)^2 = 1 + \frac{k}{M}\,\phi_1^2(\tfrac{\ell}{3})\frac{1}{\omega_1^2}$$

For simply supported beam $\phi_1(x) = \sqrt{2}\,\sin\frac{\pi x}{\ell}$

$$\therefore\ \phi_1^2(\tfrac{\ell}{3}) = 2\sin^2\frac{\pi}{3} = 1.50 \qquad \omega_1^2 = \pi^4\frac{EI}{M\ell^3} \text{ for unconstrained beam.}$$

$$\therefore\ \left(\frac{\omega}{\omega_1}\right)^2 = 1 + 1.5\left(\frac{k}{M}\right)\left(\frac{M\ell^3}{\pi^4 EI}\right)$$

11-16

From Eq. 11.2-8

$$\bar{q}_1 = \frac{1}{M(\omega_1^2-\omega^2)}\left\{-k\,\phi_1(\tfrac{\ell}{3})\left[\bar{q}_1\,\phi_1(\tfrac{\ell}{3}) + \bar{q}_2\,\phi_2(\tfrac{\ell}{3})\right]\right\}$$

$$\bar{q}_2 = \frac{1}{M(\omega_2^2-\omega^2)}\left\{-k\,\phi_2(\tfrac{\ell}{3})\left[\bar{q}_1\,\phi_1(\tfrac{\ell}{3}) + \bar{q}_2\,\phi_2(\tfrac{\ell}{3})\right]\right\}$$

Freq. Eq. becomes

$$\begin{vmatrix} \left[(\omega_1^2-\omega^2) + \frac{k}{M}\,\phi_1^2(\tfrac{\ell}{3})\right] & \frac{k}{M}\,\phi_1(\tfrac{\ell}{3})\,\phi_2(\tfrac{\ell}{3}) \\ \frac{k}{M}\,\phi_1(\tfrac{\ell}{3})\,\phi_2(\tfrac{\ell}{3}) & \left[(\omega_2^2-\omega^2) + \frac{k}{M}\,\phi_2^2(\tfrac{\ell}{3})\right] \end{vmatrix} = 0$$

11-17

We use here a somewhat different procedure than that of the text in that the influence coefs. are found from the beam eq. $\quad F(a,t) = -k\,y(\tfrac{\ell}{3})$

$$\alpha(a,x) = \alpha(\tfrac{\ell}{3},\tfrac{\ell}{3}) = \frac{(\frac{\ell}{3})(\frac{2\ell}{3})}{6EI\ell}\left(\ell^2 - \frac{4\ell^2}{9} - \frac{\ell^2}{9}\right) = \frac{4}{243}\frac{\ell^3}{EI}$$

From Eq. 11.4-4

$$y(\tfrac{\ell}{3}) = -k\,y(\tfrac{\ell}{3})\frac{4}{243}\frac{\ell^3}{EI} + \left(\frac{\omega}{\omega_1}\right)^2 q_1\,\phi_1(\tfrac{\ell}{3}) + \left(\frac{\omega}{\omega_2}\right)^2 q_2(\tfrac{\ell}{3}) + \cdots$$

Using 1st mode only

$$y(\tfrac{\ell}{3})\left[1 + \frac{4}{243}\frac{k\ell^3}{EI}\right] = \left(\frac{\omega}{\omega_1}\right)^2 q_1\,\phi_1(\tfrac{\ell}{3})$$

but $y(\tfrac{\ell}{3}) = \phi_1(\tfrac{\ell}{3})\,q_1$ + higher modes which are neglected.

$$\therefore\ \left(\frac{\omega}{\omega_1}\right)^2 = 1 + \frac{4}{243}\frac{k\ell^3}{EI}$$

Comparison with Prob. , -15

$$\frac{1.5}{\pi^4} = 0.0154$$

$$\frac{4}{243} = 0.0165$$

11-18 From Eq 11.3-4 with 1 mode

$$\phi_1(a)\ddot{\bar{q}}_1(t) = F(a,t)\,\alpha(a,a) - \left(\frac{\omega}{\omega_1}\right)^2 \bar{q}_1(t)\,\phi_1(a)$$

$$\phi_1(a)\bar{q}_1(t) = -k\,\phi_1(a)\,\bar{q}_1(t)\,\frac{\phi_1^2(a)}{M\omega_1^2} - \left(\frac{\omega}{\omega_1}\right)^2 \bar{q}_1(t)\,\phi_1(a)$$

$$\therefore \quad \left(\frac{\omega}{\omega_1}\right)^2 = 1 + \frac{k}{M\omega_1^2}\phi_1^2(a)$$

11-19

$$\bar{q}_1(\omega_1^2 - \omega^2) = -\frac{K}{M}\varphi_1'(0)\left[\bar{q}_1\varphi_1'(0) + \bar{q}_2\varphi_2'(0)\right]$$

$$\bar{q}_2(\omega_2^2 - \omega^2) = -\frac{K}{M}\varphi_2'(0)\left[\bar{q}_1\varphi_1'(0) + \bar{q}_2\varphi_2'(0)\right]$$

Freq. eq. becomes

$$\begin{vmatrix} \left[(\omega_1^2-\omega^2) + \frac{K}{M}\varphi_1'^2(0)\right] & \frac{K}{M}\varphi_1'(0)\varphi_2'(0) \\ \frac{K}{M}\varphi_1'(0)\varphi_2'(0) & \left[(\omega_2^2-\omega^2) + \frac{K}{M}\varphi_2'(0)\right] \end{vmatrix} = 0$$

$\omega_2^2 = 16\,\omega_1^2$ for simply supported beam. Let $\lambda = \left(\frac{\omega}{\omega_1}\right)^2$

$$\left[(1-\lambda) + \frac{K}{M\omega_1^2}\varphi_1'^2(0)\right]\left[(16-\lambda) + \frac{K}{M\omega_1^2}\varphi_2'^2(0)\right] - \left(\frac{K}{M\omega_1^2}\right)^2\left[\varphi_1'(0)\,\varphi_2'(0)\right]^2 = 0$$

$$\lambda^2 - \left\{17 + \frac{K}{M\omega_1^2}\left[\varphi_1'^2(0) + \varphi_2'^2(0)\right]\right\}\lambda + \left\{16 + \frac{K}{M\omega_1^2}\left[\varphi_2'^2(0) + 16\,\varphi_1'^2(0)\right]\right\} = 0$$

$$\varphi_1(x) = \sqrt{2}\sin\frac{\pi x}{\ell} \qquad \varphi_1'(0) = \sqrt{2}\,\frac{\pi}{\ell} \qquad \varphi_1'^2(0) = 2\left(\frac{\pi}{\ell}\right)^2$$

$$\varphi_2(x) = \sqrt{2}\sin\frac{2\pi x}{\ell} \qquad \varphi_2'(0) = 2\sqrt{2}\,\frac{\pi}{\ell} \qquad \varphi_2'^2(0) = 8\left(\frac{\pi}{\ell}\right)^2$$

If $K = 0$

$$\lambda^2 - 17\lambda + 16 = (\lambda - 1)(\lambda - 16) = 0$$

$$\therefore \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 16 \end{cases}$$

If $K \neq 0$ Let $\alpha = \frac{K}{M\omega_1^2}$

$$\lambda^2 - \left[17 + 10\left(\frac{\pi}{\ell}\right)^2\alpha\right]\lambda + \left[16 + 40\left(\frac{\pi}{\ell}\right)^2\alpha\right] = 0$$

11-21 Cont:

$$T = \int_0^l m\varphi_0^2 \dot{q}_0^2\, dm + \int_0^l 2m\varphi_0\varphi_1 \dot{q}_0 \dot{q}_1\, dx + \int_0^l m\varphi_1^2 \dot{q}_1^2\, dx + \frac{1}{2} M_0 \varphi_0^2 \dot{q}_0^2$$

$$= \frac{1}{2}\dot{q}_0^2 \left\{ M_0 + 2ml \right\} + \left\{ 2ml \cdot .783 \right\} \dot{q}_0 \dot{q}_1 + ml\, \dot{q}_1^2$$

$$U = \frac{1}{2}\int_0^l 2EI\, y''^2(x,t)\, dx = \frac{1}{2}\left\{ 2\omega_1^2\, ml\, q_1^2 \right\} \qquad \text{see Eq. 11.1-7}$$

Lagranges Eqs.

$$\ddot{q}_0 (M_0 + 2ml) + 2 \times .783\, ml\, \ddot{q}_1 = 0$$

$$\ddot{q}_1\, 2ml + 2 \times .783\, ml\, \ddot{q}_0 + 2\omega_1^2 ml\, q_1 = 0$$

Freq. Eq

$$\begin{vmatrix} -(M_0 + 2ml)\omega^2 & -2 \times .783\, ml\,\omega^2 \\ -2 \times .783\, ml\,\omega^2 & 2(\omega_1^2 - \omega^2)\, ml \end{vmatrix} = 0$$

Let $2ml = M$

$$-(M_0 + M)\omega^2(\omega_1^2 - \omega^2)M - (.783)^2 M^2 \omega^4 = 0$$

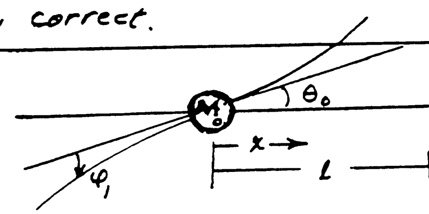
$$\frac{\omega}{\omega_1} = \sqrt{\frac{M_0 M + M^2}{M_0 M + .387 M^2}}$$ where $\omega_1 = 1^{st}$ nat. freq. of cantilever beam of length l and mass $\frac{1}{2} M$

If $M_0 \to 0$, then $\omega = \omega_1 \dfrac{1}{\sqrt{.387}} = 1.61\, \omega_1$

Since $\omega_1 = 3.52\sqrt{\dfrac{EI}{ml^4}}$ and for free-free beam of length $2l$ has nat. freq $\omega_{1_{ff}} = 22.4\sqrt{\dfrac{EI}{m(2l)^4}} = 5.57\sqrt{\dfrac{EI}{ml^4}}$

$\dfrac{5.57}{3.52} = 1.58$, Since the airplane becomes a free-free beam of length $2l$ as $M_0 \to 0$, the above result is approximately correct.

11-22



$$y(x,t) = q_0 \frac{x}{l} + \varphi_1 q_1$$

$$\theta_0 = \left(\frac{dy}{dx}\right)_{x=0} = \frac{q_0}{l}$$

$$T = \frac{1}{2} I_0 \left(\frac{\dot{q}_0}{l}\right)^2 + \frac{1}{2}\int_0^l 2m\left[\dot{q}_0 \frac{x}{l} + \varphi_1 \dot{q}_1\right]^2 dx$$

$$= \frac{1}{2} I_0 \left(\frac{\dot{q}_0}{l}\right)^2 + \frac{1}{3} ml\, \dot{q}_0^2 + 2m\dot{q}_0 \dot{q}_1 \int_0^l \frac{x}{l}\varphi_1(x)\, dx + m\dot{q}_1^2 \int_0^l \varphi_1^2\, dx$$

11-22 Cont:

$$U = \frac{1}{2}\left\{ 2\omega_1^2 (m\ell) q_1^2 \right\}$$ see Eq. 11.1-7

Lagranges Eq. & Characteristic eq.

$$\begin{vmatrix} -\left(\frac{I_o}{\ell^2} + \frac{2m\ell}{3}\right)\omega^2 & -\frac{2m\omega^2}{\ell}\int_0^\ell x\varphi_1\,dx \\ -\frac{2m\omega^2}{\ell}\int_0^\ell x\varphi_1\,dx & 2m\left(\ell\omega_1^2 - \omega^2\int_0^\ell \varphi_1^2\,dx\right) \end{vmatrix} = 0$$

For cantilever $\int_0^\ell \varphi_1^2\,dx = \ell$, $\int_0^\ell x\varphi_1\,dx = \frac{2}{\beta_1^2} = \frac{2\ell}{3.516} = \frac{\ell^2}{1.758}$

$$\omega^4\left[2m\ell\left(\frac{I_o}{\ell^2} + \frac{2}{3}m\ell\right) - \frac{(2m\ell)^2}{3.08}\right] = \omega^2\omega_1^2\, 2m\ell\left(\frac{I_o}{\ell^2} + \frac{2}{3}m\ell\right)$$

$$\omega = \omega_1\sqrt{\frac{\left(I_o + \frac{2}{3}m\ell^3\right)}{\left(I_o + \frac{2}{3}m\ell^3\right) - \frac{2m\ell^3}{3.090}}}$$

To check for case $I_o = 0$, the results should be the 2nd mode of a hinged-free beam of Length ℓ or 2nd mode of free-free beam of Length 2ℓ.

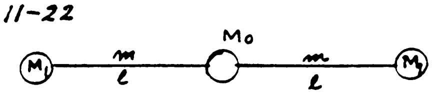
For $I_o = 0$, $\omega = \omega_1\sqrt{\frac{3.09}{.090}} = \omega_1\sqrt{34.5} = 5.85\,\omega_1$

$= 20.6\sqrt{\frac{EI}{m\ell^4}}$ * However correct result should be $15.4\sqrt{\frac{EI}{m\ell^4}}$

*Difficulty due to difference of two numbers in denominator which is small. Needs 2nd mode φ_2 for better results.

11-23 See Prob. 11-21 and 11-22

T has added term $2(\frac{1}{2})M_1\dot{y}^2(\ell,t)$



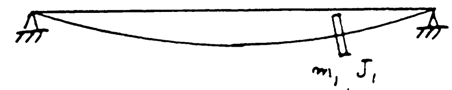
$$= M_1\left[\varphi_0\dot{q}_0 + \varphi_1(\ell)\dot{q}_1\right]^2 = M_1\varphi_0^2\dot{q}_0^2 + 2M_1\varphi_0\varphi(\ell)\dot{q}_0\dot{q}_1 + M_1\varphi_1^2(\ell)\dot{q}_1^2$$

Lagrange's Eq. same as Prob 11-21 $+ 2M_1\left[\ddot{q}_0 + \varphi_1(\ell)\ddot{q}_1\right] + 2M_1\left[\varphi_1(\ell)\ddot{q}_0 + \varphi_1^2(\ell)\ddot{q}_1\right]$

New freq. eq.

$$\begin{vmatrix} -(M_o + 2m\ell + 2M_1)\omega^2 & -(2\times.783m\ell + 2M_1\varphi_1(\ell))\,\omega^2 \\ -(2\times.783m\ell + 2M_1\varphi_1(\ell))\,\omega^2 & \left[2m\ell\omega_1^2 - \omega^2(2m\ell + 2M_1\varphi_1^2(\ell))\right] \end{vmatrix} = 0$$

11-24



The additional mass changes only $T = \frac{1}{2} m_1 \dot{y}^2 + \frac{1}{2} J_1 \dot{y}'^2 + \frac{1}{2}\int \dot{y}^2 dm$

$$y(x,t) = \varphi_1(x)\, q_1(t)$$

$$T = \frac{1}{2} m_1 \varphi_1^2(a)\dot{q}_1^2 + \frac{1}{2} J_1 \varphi_1'^2(a)\dot{q}_1^2 + \frac{1}{2}\int \varphi_1^2(x)\,dm \cdot \dot{q}_1^2$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_1} = m_1 \varphi_1^2(a)\ddot{q}_1 + J_1 \varphi_1'^2(a)\ddot{q}_1 + \ddot{q}_1 \int_0^\ell \varphi_1^2(x)\,dx$$

$$= \left[M_1 + m_1 \varphi_1^2(a) + J_1 \varphi_1'^2(a) \right] \ddot{q}_1$$

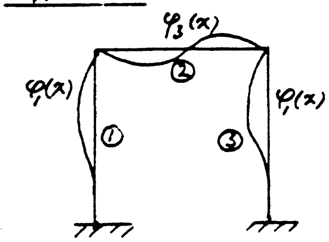
$$= M_1\left[1 + \frac{m_1}{M_1}\varphi_1^2(a) + \frac{J_1}{M_1}\varphi_1'^2(a)\right]\ddot{q}_1 = M'\ddot{q}_1$$

$$M'\ddot{q}_1 + C_1\dot{q}_1 + K_1 q_1 = \int_0^\ell p(x,t)\,\varphi_1(x)\,dx$$

$$\therefore \frac{C_1}{M_1} = 2\zeta'\omega_1 = \frac{C_1}{M_1\left[1 + \frac{m_1}{M_1}\varphi_1^2(a) + \frac{J_1}{M_1}\varphi_1'^2(a)\right]}$$

$$\frac{K_1}{M'} = \omega'^2 = \frac{K}{\text{same denominator}} \qquad \text{etc.}$$

11-25



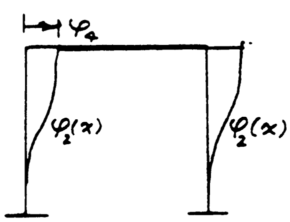
The modes are available in App. D.

Member ① $w_1(x) = \varphi_1 p_1 + \varphi_2 p_2$

Member ② $w_2(x) = \varphi_3 p_3$

$u_2(x) = 1\, p_4$

Member ③ $w_3(x) = \varphi_1 p_5 + \varphi_2 p_6$



So there are 6 coordinates $p_1 \cdots p_6$

Boundary conditions (4 eqs.)

$w_1(\ell) = u_2(0) \qquad \varphi_1(\ell)p_1 + \varphi_2(\ell)p_2 = p_4$

$w_1'(\ell) = w_2'(0) \qquad \varphi_1'(\ell)p_1 + \varphi_2'(\ell)p_2 = \varphi_3'(0)p_3$

$w_2'(\ell) = w_3'(\ell) \qquad \varphi_3'(\ell)p_3 = \varphi_1'(\ell)p_5 + \varphi_2'(\ell)p_6$

$u_2(\ell) = w_3(\ell) \qquad p_4 = \varphi_1'(\ell)p_5 + \varphi_2'(\ell)p_6$

11-25 Cont.: Gen. mass & gen. stiffness from T & U

$$T = \frac{1}{2}\int_0^{l_1} \dot{w}_1^2 \, m \, dx + \frac{1}{2}\int_0^{l_2} [\dot{w}_2^2 + \dot{u}_2^2] \, m \, dx + \frac{1}{2}\int_0^{l_1} \dot{w}_3^2 \, m \, dx$$

$$U = \frac{1}{2} EI \int_0^{l_1} w_1''^2 dx + \frac{1}{2} EI \int_0^{l_2} w_2''^2 dx + \frac{1}{2} EI \int_0^{l_1} w_3''^2 dx$$

Mass Matrix $\begin{bmatrix} m_{11} & m_{12} & & & & \\ m_{21} & m_{22} & & & & \\ & & m_{33} & m_{34} & & \\ & & m_{43} & m_{44} & & \\ & & & & m_{55} & m_{56} \\ & & & & m_{65} & m_{66} \end{bmatrix}$ Stiffness Matrix $\begin{bmatrix} k_{11} & k_{12} & & & & \\ k_{21} & k_{22} & & & & \\ & & k_{33} & k_{34} & & \\ & & k_{43} & k_{44} & & \\ & & & & k_{55} & k_{56} \\ & & & & k_{65} & k_{66} \end{bmatrix}$

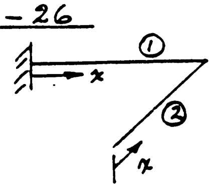
Constraint Eqs.

$$\begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{Bmatrix} = \begin{bmatrix} \\ 2\times 6 \\ \text{Matrix} \\ \\ \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

where q_1 & q_2 may be any of the p_s

System reduces to a 2×2 matrix equation.

11-26



Let $w_1(x) = \phi_1 p_1 + \phi_2 p_2 + \phi_3 p_3$

$\theta_1(x) = \phi_4 p_4$

$w_2(x) = \phi_5 p_5 + \phi_6 p_6 + \phi_7 p_7$

$\theta_2(x) = \phi_8 p_8 + \phi_9 p_9$

Subst. into

$$T = \frac{1}{2}\int \dot{w}^2 dm + \frac{1}{2}\int \frac{J}{A} \dot{\theta}^2 dm$$

$$U = \frac{1}{2}\int EI \left(\frac{d^2 w}{dx^2}\right)^2 dx + \frac{1}{2}\int C \left(\frac{d\theta}{dx}\right)^2 dx$$

to establish m_{ij} and k_{ij}

11-26 Cont: Constraint eq. at junction of 1 and 2

(1)	$w_1(\ell) = w_2(\ell)$	defl 1 = defl 2
(2)	$\theta_1(\ell) = -w_2'(\ell)$	twist 1 = bending slope 2
(3)	$C\,\theta_1'(\ell) = EI\,w_2''(\ell)$	Torque 1 = bend. moment 2
(4)	$w_1'''(\ell) = -w_2'''(\ell)$	shear 1 = shear 2
(5)	$EI\,w_1''(\ell) = -C\,\theta_2'(\ell)$	bend. mom. 1 = torque at 2
(6)	$w_1'(\ell) = \theta_2(\ell)$	bend slope 1 = twist 2

Mass Matrix

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} & & & & & & \\ m_{21} & m_{22} & m_{23} & & & \text{all } 0 & & & \\ m_{31} & m_{32} & m_{33} & & & & & & \\ & & & m_{44} & 0 & 0 & 0 & 0 & 0 \\ & & & & m_{55} & m_{56} & m_{57} & 0 & 0 \\ & & & & m_{65} & m_{66} & m_{67} & 0 & 0 \\ & \text{all } 0 & & & m_{75} & m_{76} & m_{77} & 0 & 0 \\ & & & & & & & m_{88} & m_{89} \\ & & & & & & & m_{98} & m_{99} \end{bmatrix} = 9\times 9$$

Stiffness Matrix

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & & & & & & \\ k_{21} & k_{22} & k_{23} & & & & & & \\ k_{31} & k_{32} & k_{33} & & & & & & \\ & & & k_{44} & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & k_{77} & & \\ & & & & & & & & \\ & & & & & & & & k_{99} \end{bmatrix} = 9\times 9$$

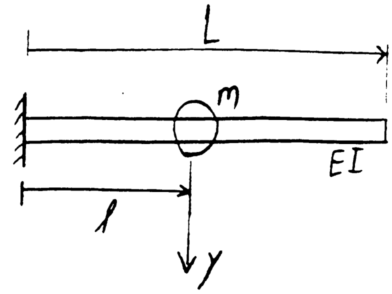
Constraint Matrix $= 9 \times 3$ $\begin{bmatrix} \vdots & \vdots & \vdots \end{bmatrix}$

3 gen. coords, any 3 of the P_s $\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$

Result is

$$\begin{bmatrix} M \\ 3\times 3 \end{bmatrix}\begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{Bmatrix} + \begin{bmatrix} K \\ 3\times 3 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \{0\}$$

11-27

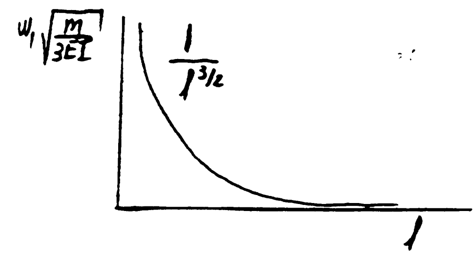


From the table of spring stiffness in chapter 2 at a distance l, $k = 3EI/l^3$

$$\therefore \quad mg = ky \Rightarrow y = mgl^3/3EI$$

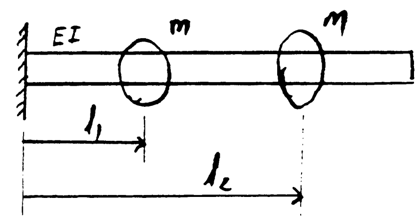
$$\Rightarrow \quad w_1^2 = \frac{gmy}{my^2} = \frac{g}{y} = \frac{3EI}{ml^3} \quad \left(= \frac{k}{m}\right)$$

$$\therefore \quad w_1 = \sqrt{3EI/ml^3}$$



11-28

The natural frequences due to each mass are (see problem 11-27)

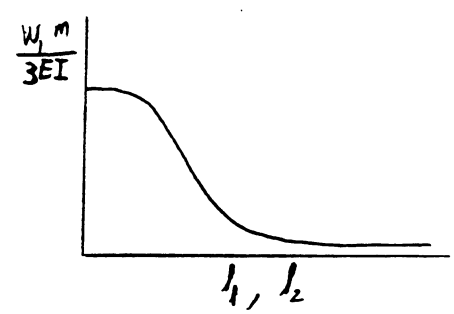


$$w_{11}^2 = 3EI/ml_1^3 \quad \text{and} \quad w_{22}^2 = 3EI/ml_2^3$$

Using Dunkerley's equation, we get

$$w_1^2 = \frac{w_{11}^2 w_{22}^2}{w_{11}^2 + w_{22}^2} = \left(\frac{3EI}{m}\right)^2 \frac{1/l_1^6 l_2^6}{\frac{1}{l_1^6} + \frac{1}{l_2^6}}$$

$$\therefore \quad w_1 = \frac{3EI}{m}\sqrt{\frac{1}{l_1^6 + l_2^6}}$$



11-29

For problem 11-27 we have

$k = \frac{3EI}{l^3}$

$\therefore \quad k - w^2 m = 0$

$$\Rightarrow \quad w^2 = \frac{k}{m} = \frac{3EI}{ml^3} \Rightarrow w = \sqrt{\frac{3EI}{ml^3}}$$

For problem 11-28 we have two nodes, one at l_1 and the other at l_2

The mass matrix is $M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$

Let the deflections of the two masses be y_1 and y_2. To find the stiffness matrix, we write

$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, where F_1 and F_2 are applied at the two masses

let $y_1 = 1$ and $y_2 = 0 \Longrightarrow F_1$

12-1

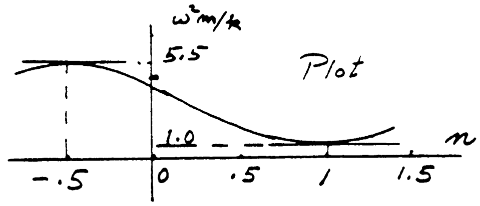
$$T = \tfrac{1}{2} m \dot{x}_1^2 + \tfrac{1}{2}(2m)\dot{x}_2^2 \qquad U = \tfrac{1}{2} k x_1^2 + \tfrac{1}{2}(3k)(x_2 - x_1)^2 + \tfrac{1}{2}(2k)x_2^2$$

Let $\dot{x} = \omega x$ & equate

$$\omega^2 m x_1^2 + 2\omega^2 m x_2^2 = k x_1^2 + 3k(x_2 - x_1)^2 + 2k x_2^2$$

Let $x_2/x_1 = n$, then

$$\frac{\omega^2 m}{k} = \frac{1 + 3(n-1)^2 + 2n^2}{1 + 2n^2}$$

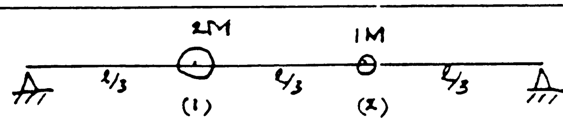


Can be checked by

$$\frac{\partial}{\partial n}\left(\frac{\omega^2 m}{k}\right) = 0 \;\rightarrow\; \text{gives} \quad n = -.5 \quad \text{and} \quad n = 1.0$$

Roots are $\frac{\omega^2 m}{k} = 1.0$ and $\frac{\omega^2 m}{k} = 5.50$

12-2



$$y(x) = \frac{Wbx}{6lEI}(l^2 - x^2 - b^2) \qquad 0 < x < (l - b)$$

Due to 2M

$$y_1 = \frac{(2Mg)\frac{2}{3}l\,\frac{1}{3}l}{6lEI}\left[l^2 - \left(\frac{l}{3}\right)^2 - \left(\frac{2l}{3}\right)^2\right] = \frac{16}{486}\left(\frac{Mgl^3}{EI}\right)$$

$$y_2 = \frac{(2Mg)\frac{1}{3}l\,\frac{1}{3}l}{6l\,EI}\left[l^2 - \left(\frac{l}{3}\right)^2 - \left(\frac{l}{3}\right)^2\right] = \frac{14}{486}\left(\frac{Mgl^3}{EI}\right)$$

Due to 1M

$$y_1 = \frac{Mg\,\frac{1}{3}l\,\frac{1}{3}l}{6l\,EI}\left[l^2 - \left(\frac{l}{3}\right)^2 - \left(\frac{l}{3}\right)^2\right] = \frac{7}{486}\left(\frac{Mgl^3}{EI}\right)$$

$$y_2 = \frac{Mg\,\frac{1}{3}l\,\frac{2}{3}l}{6l\,EI}\left[l^2 - \left(\frac{2}{3}l\right)^2 - \left(\frac{l}{3}\right)^2\right] = \frac{8}{486}\left(\frac{Mgl^3}{EI}\right)$$

Add $y_1 = \frac{23}{486}\left(\frac{Mgl^3}{EI}\right) \qquad y_2 = \frac{22}{486}\left(\frac{Mgl^3}{EI}\right)$

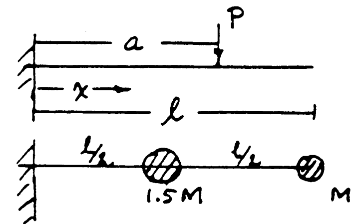
$$\omega_1^2 = \frac{g\left[(2M)\frac{23}{486} + (M)\frac{22}{486}\right]}{\left[2M\left(\frac{23}{486}\right)^2 + M\left(\frac{22}{486}\right)^2\right]}\;\frac{1}{\frac{Mgl^3}{EI}} = 21.43\,\frac{EI}{Ml^3}$$

$$\omega_1 = 4.63\sqrt{\frac{EI}{Ml^3}} \quad \text{rad/s.}$$

12-3

$$y(x) = \frac{Px^2}{6EI}(3a - x) \qquad 0 \le x \le a$$

$$= \frac{Pa^2}{6EI}(3x - a) \qquad a \le x \le \ell$$



Due to 1.5M & M with results superimposed (1) (2)

$$y_1 = \frac{1.5Mg\left(\frac{\ell}{2}\right)^3}{3EI} + \frac{Mg\left(\frac{\ell}{2}\right)^2}{6EI}\left(3\ell - \frac{\ell}{2}\right) = \frac{16}{96}\frac{Mg\ell^3}{EI}$$

$$y_2 = \frac{1.5Mg\left(\frac{\ell}{2}\right)^2}{6EI}\left(3\ell - \frac{\ell}{2}\right) + \frac{Mg\,\ell^3}{3EI} = \frac{47}{96}\frac{Mg\ell^3}{EI}$$

$$\omega^2 = \frac{\cancel{g}\left[1.5\left(\frac{16}{96}\right) + \left(\frac{47}{96}\right)\right]}{\left[1.5\left(\frac{16}{96}\right)^2 + \left(\frac{47}{96}\right)^2\right]}\frac{EI}{M\cancel{g}\ell^3} = 2.628\frac{EI}{M\ell^3}$$

$$\therefore \omega_1 = 1.621\sqrt{\frac{EI}{M\ell^3}}$$

12-4

$$a_{11} = \frac{1}{24}\frac{\ell^3}{EI} \qquad a_{21} = a_{12} = \frac{10}{96}\frac{\ell^3}{EI}$$

$$a_{22} = \frac{\ell^3}{3EI} \qquad \therefore [a] = \frac{\ell^3}{96EI}\begin{bmatrix} 4 & 10 \\ 10 & 32 \end{bmatrix}$$

$$[k] = [a]^{-1} = \frac{96EI}{\ell^3}\frac{1}{28}\begin{bmatrix} 32 & -10 \\ -10 & 4 \end{bmatrix} = 13.7143\frac{EI}{\ell^3}\begin{bmatrix} 8 & -2.5 \\ -2.5 & 1 \end{bmatrix}$$

Set up eq. of motion using flexibility

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \frac{\omega^2 M\ell^3}{96EI}\begin{bmatrix} 4 & 10 \\ 10 & 32 \end{bmatrix}\begin{bmatrix} 1.5 & 0 \\ 0 & 1.0 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$$

$$= \frac{\omega^2 M\ell^3}{96EI}\begin{bmatrix} 6 & 10 \\ 15 & 32 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \qquad \text{Let } \lambda = \frac{\omega^2 M\ell^3}{96EI}$$

$$\begin{bmatrix} (1 - 6\lambda) & -10\lambda \\ -15\lambda & (1 - 32\lambda) \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix}$$

12-4 Cont.

$$42\lambda^2 - 38\lambda + 1 = 0$$

$$\lambda^2 - 0.9048\lambda + 0.0238 = 0$$

$$\lambda = 0.4524 \pm \sqrt{.2048 - .0238} = 0.4524 \pm 0.4253$$

$$\lambda = \frac{\omega^2 M\ell^3}{96EI} = \begin{cases} 0.0271 \\ 0.8771 \end{cases} \qquad \omega^2 = \begin{cases} 2.6016 \ \frac{EI}{M\ell^3} \\ 84.2576 \ \frac{EI}{M\ell^3} \end{cases}$$

$$\omega = \begin{cases} 1.6129 \sqrt{\frac{EI}{M\ell^3}} \\ 9.1792 \sqrt{\frac{EI}{M\ell^3}} \end{cases} \qquad \left(\frac{X_1}{X_2}\right)_1 = \frac{10\lambda}{1 - 6\lambda} = 0.3237$$

To verify $\omega^2 = \dfrac{X'KX}{X'MX}$

$$\omega^2 = \frac{(.3237 \quad 1.0)\begin{bmatrix} 8 & -2.5 \\ -2.5 & 1 \end{bmatrix}\begin{Bmatrix} .3237 \\ 1.0 \end{Bmatrix} 13.7143 \frac{EI}{\ell^3}}{(.3237 \quad 1.0)\begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} .3237 \\ 1.0 \end{Bmatrix}} = 2.605 \frac{EI}{M\ell^3}$$

$$\therefore \omega_1 = 1.614 \sqrt{\frac{EI}{M\ell^3}}$$

12-5

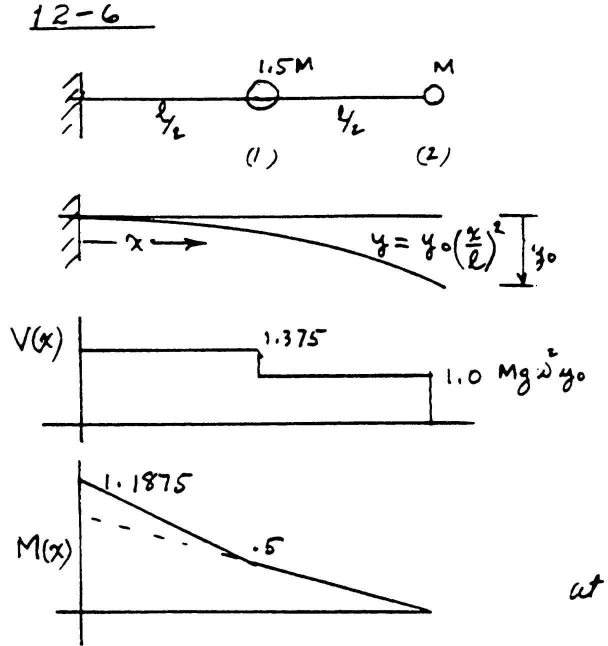
$$\omega^2 = \frac{X'MX}{X'M'aMX} =$$

$$= \frac{(.3237 \quad 1.0)\begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} .3237 \\ 1.0 \end{Bmatrix} \quad \frac{96EI}{M\ell^3}}{(.3237 \quad 1.0)\begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 10 \\ 10 & 32 \end{bmatrix}\begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} .3237 \\ 1.0 \end{Bmatrix}}$$

$$= \frac{1.1572}{42.654} \cdot \frac{96EI}{M\ell^3} = 2.6045 \frac{EI}{M\ell^3}$$

$$\omega_1 = 1.6138 \sqrt{\frac{EI}{M\ell^3}}$$

12-6



$\dot{y} = \omega y = \omega y_0 \left(\frac{x}{l}\right)^2$

$\ddot{y} = -\omega^2 y = -\omega^2 y_0 \left(\frac{x}{l}\right)^2$

$-m\ddot{y} = \omega^2 y_0 \left(\frac{x}{l}\right)^2 m$

For lumped mass, dynamic loads are

$\omega^2 M y_0 g$ at (2)

and $\omega^2 1.5 M y_0 g \left[\frac{1}{4}\right] + \omega^2 M y_0 g [1]$

at (1) $= \omega^2 M g y_0 \; 1.375$

$$\therefore V(x) = Mg\omega^2 y_0 \qquad \frac{l}{2} \le x \le l$$

$$= 1.375\, Mg\omega^2 y_0 \qquad 0 \le x \le \frac{l}{2}$$

$$M(x) = \int_x^l V(\xi)\,d\xi = Mg\omega^2 y_0 \,\xi \Big|_x^l = Mg\omega^2 y_0 (l - x) \qquad \frac{l}{2} \le x \le l$$

$$= Mg\omega^2 y_0 \left[\frac{l}{2} + 1.375\left(\frac{l}{2} - x\right)\right] \qquad 0 \le x \le \frac{l}{2}$$

$$U = \frac{1}{2}\int \frac{M^2}{EI}dx = \frac{(Mg\omega^2 y_0)^2}{2EI}\left[\int_{l/2}^{l} (l-x)^2 dx + \int_0^{l/2} \left\{\frac{l}{2} + 1.375\left(\frac{l}{2} - x\right)\right\}^2 dx\right]$$

$$= \frac{1}{2}\frac{(Mg\omega^2 y_0)^2 l^3}{EI}\left[\frac{1}{24} + \frac{9.0156}{24}\right] = \frac{1}{2}\frac{M^2 g^2 \omega^4 y_0^2}{EI}(0.4173)$$

$$T = \frac{1}{2} M \omega^2 y_0^2 + \frac{1}{2}(1.5M)\omega^2 y_0^2 \left(\frac{1}{2}\right)^4 = \frac{1}{2} M\omega^2 y_0^2 (1.0938)$$

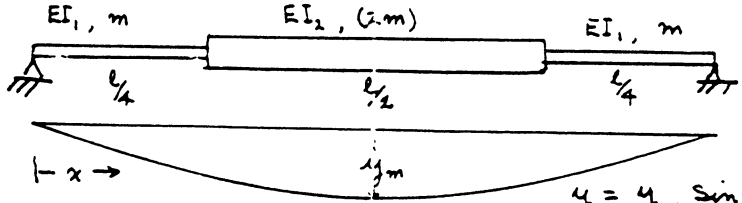
12-6 Cont

$U = T$ gives

$$\omega^2 = \frac{1.0938}{0.4173}\frac{EI}{M\ell^3} = 2.619\frac{EI}{M\ell^3}$$

$$\omega_1 = 1.618\sqrt{\frac{EI}{M\ell^3}} \qquad \text{Exact val} = 1.6129\sqrt{\frac{EI}{M\ell^3}}$$

12-7



$$y = y_m \sin\frac{\pi x}{\ell}$$

$$\frac{d^2y}{dx^2} = -y_m\left(\frac{\pi}{\ell}\right)^2 \sin\frac{\pi x}{\ell}$$

$$U = \frac{1}{2}\int EI\left(\frac{d^2y}{dx^2}\right)^2 dx = \frac{1}{2}y_m^2\left(\frac{\pi}{\ell}\right)^4\left[EI_1\cdot 2\int_0^{\ell/4}\sin^2\frac{\pi x}{\ell}dx + EI_2\, 2\int_{\ell/4}^{\ell/2}\sin^2\frac{\pi x}{\ell}dx\right]$$

$$= y_m^2\left(\frac{\pi}{\ell}\right)^4\left[EI_1\,\frac{1}{2}\left(x - \frac{\ell}{2\pi}\sin\frac{2\pi x}{\ell}\right)_0^{\ell/4} + EI_2\,\frac{1}{2}\left(x - \frac{\ell}{2\pi}\sin\frac{2\pi x}{\ell}\right)_{\ell/4}^{\ell/2}\right]$$

$$= y_m^2\left(\frac{\pi}{\ell}\right)^4\ell\left[.0454\,EI_1 + .2046\,EI_2\right]$$

$$T = \frac{1}{2}\int m\dot{y}^2 dx = \frac{1}{2}\omega^2 y_m^2\left[m\, 2\int_0^{\ell/4}\sin^2\frac{\pi x}{\ell}dx + (2m)\, m\int_{\ell/4}^{\ell/2}\sin^2\frac{\pi x}{\ell}dx\right]$$

$$= \omega^2 y_m^2\, m\ell\left[.0454 + .4092\right] = 0.4546\,\omega^2 y_m^2\, m\ell$$

Equating

$$\omega_1^2 = \frac{\pi^4}{m\ell^4}\left[0.100\,EI_1 + 0.450\,EI_2\right]$$

$$\omega_1 = \pi^2\sqrt{\frac{0.100\,EI_1 + 0.450\,EI_2}{m\ell^4}}$$

12-7 Cont:

(a) if $EI_2 = EI_1$ & $m_1 = m$, $m_2 = 2m$

$$\omega_1 = .7416\,\pi^2 \sqrt{\frac{EI_1}{m_1 \ell^4}} = 7.32 \sqrt{\frac{EI_1}{m\ell^4}}$$

(b) if $EI_2 = 4EI_1$

$$\omega_1 = 1.3785\,\pi^2 \sqrt{\frac{EI_1}{m\ell^4}}$$

12-8 $y = y_0 \frac{x}{\ell}\left(1 - \frac{x}{\ell}\right)$ $\frac{dy}{dx} = \frac{y_0}{\ell}\left(1 - 2\frac{x}{\ell}\right)$

$$\frac{d^2y}{dx^2} = -y_0 \frac{2}{\ell^2}$$

$$U = \frac{1}{2} \frac{4y_0^2}{\ell^4}\left[2EI_1\int_0^{\ell/4} dx + 2EI_2 \int_{\ell/4}^{\ell/2} dx\right] = \frac{1}{2}\left(\frac{4y_0^2}{\ell^4}\right)\frac{\ell}{2}\left[EI_1 + EI_2\right]$$

$$= \frac{y_0^2}{\ell^3}\left[EI_1 + EI_2\right]$$

$$T = \cancel{\tfrac{1}{2}}\,\omega^2 y_0^2\left[m\cancel{2}\int_0^{\ell/4}\left(\frac{x}{\ell}\right)^2\left(1 - \frac{x}{\ell}\right)^2 dx + (2m)\cancel{2}\int_{\ell/4}^{\ell/2}\left(\frac{x}{\ell}\right)^2\left(1 - \frac{x}{\ell}\right)^2 dx\right]$$

$$= \omega^2 y_0^2\left[m\ell\left\{\frac{1}{3}\left(\frac{x}{\ell}\right)^3 - \frac{1}{2}\left(\frac{x}{\ell}\right)^4 + \frac{1}{5}\left(\frac{x}{\ell}\right)^5\right\}_0^{\ell/4} + 2m\ell\left\{\quad \text{same} \quad\right\}_{\ell/4}^{\ell/2}\right]$$

$$= \omega^2 y_0^2\, m\ell\left[(.008659) + (.0264)\right] = .03506\,\omega^2 y_0^2\, m\ell$$

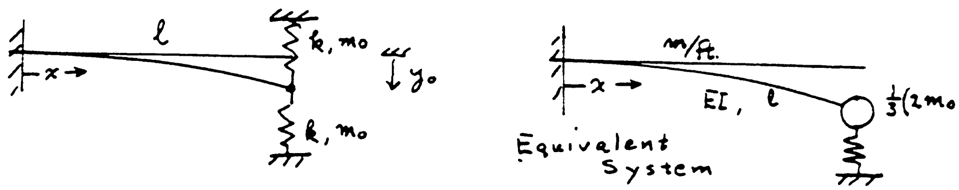
Equating $T = U$

$$\omega_1^2 = 28.52\left(\frac{EI_1 + EI_2}{m\ell^4}\right)$$

For $EI_2 = EI_1$ $\omega_1 = \sqrt{57.04}\sqrt{\frac{EI_1}{m\ell^4}} = 7.55\sqrt{\frac{EI_1}{m\ell^4}}$

for Sine Curve Prob. 12-7 $\omega_1 = 7.32\sqrt{\frac{EI_1}{m\ell^4}}$

12-9



$$T_m = \frac{1}{2}\left(\frac{33\,m\ell}{140}\right)\omega^2 y_o^2 + \frac{1}{2}\left(\frac{2}{3}m_o\right)\omega^2 y_o^2$$

$$U_m = \frac{1}{2}\left(\frac{3EI}{\ell^3}\right)y_o^2 + \frac{1}{2}(2k)\,y_o^2$$

Equate

$$\omega_1^2 = \frac{\frac{3EI}{\ell^3} + 2k}{\left(\frac{33\,m\ell}{140} + \frac{2}{3}m_o\right)}$$

12-10

$$y_m = \sin\frac{\pi x}{\ell} + b$$

$$U = \frac{1}{2}EI\int_0^\ell (y'')^2\,dx + \frac{1}{2}\left(\frac{k}{2}\right)y_{x=0}^2 + \frac{1}{2}\left(\frac{k}{2}\right)y_{x=\ell}^2$$

$$U_{max} = \frac{1}{2}EI\int_0^\ell \left(\frac{\pi}{\ell}\right)^4 \sin^2\frac{\pi x}{\ell}\,dx + \frac{1}{2}kb^2$$

$$T_{max} = \frac{1}{2}\int_0^\ell \frac{M}{\ell}\dot{y}^2\,dx = \frac{1}{2}\omega^2\frac{M}{\ell}\int_0^\ell\left(\sin^2\frac{\pi x}{\ell} + 2b\sin\frac{\pi x}{\ell} + b^2\right)dx$$

$$= \frac{M\omega^2}{2}\left(\frac{1}{2} + \frac{4b}{\pi} + b^2\right)$$

Equating $U_{max} = T_{max}$
$$\omega_1^2 = \frac{2k}{M}\left(\frac{\frac{K}{k}\frac{\pi^4}{4} + \frac{b^2}{2}}{\frac{1}{2} + \frac{4b}{\pi} + b^2}\right)$$

where $K = EI/\ell^3$

$\frac{\partial\omega_1^2}{\partial b} = 0$ gives

$$b = -\frac{\pi}{4}\left(\frac{1}{2} - \frac{K\pi^4}{2k}\right) \pm \sqrt{\left[\frac{\pi}{4}\left(\frac{1}{2} - \frac{K\pi^4}{2k}\right)\right]^2 + \frac{\pi^4 K}{2k}}$$

12-11

$$y = y_m\left[3\left(\frac{x}{l}\right) - 4\left(\frac{x}{l}\right)^3\right] \qquad 0 \le x \le \frac{l}{2}$$

$$m(x) = m_0\left(\frac{x}{l}\right)\left(1 - \frac{x}{l}\right)$$

$$T_m = \frac{1}{2}\times 2\int_0^{l/2} m(x)\,\omega^2 y^2\,dx = m_0\,\omega^2 y_m^2 \int_0^{l/2} \frac{x}{l}\left[9\left(\frac{x}{l}\right)^2 - 24\left(\frac{x}{l}\right)^4 + 16\left(\frac{x}{l}\right)^6\right] dx$$

$$= m_0\,\omega^2 y_m^2\, l\left[\frac{9}{4}\left(\frac{1}{2}\right)^4 - \frac{24}{6}\left(\frac{1}{2}\right)^6 + \frac{16}{8}\left(\frac{1}{2}\right)^8 - \frac{9}{5}\left(\frac{1}{2}\right)^5 + \frac{24}{7}\left(\frac{1}{2}\right)^7 - \frac{16}{9}\left(\frac{1}{2}\right)^9\right]$$

$$= 0.0529\ m_0 l\ \omega^2 y_m^2$$

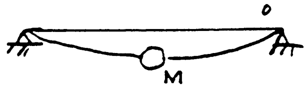
$$U_m = \frac{1}{2}\times 2\int_0^{l/2} EI\left(\frac{d^2y}{dx^2}\right)^2 dx = y_m^2\ EI\int_0^{l/2} \frac{24^2}{l^6}\,x^2\,dx = y_m^2\ EI\,\frac{24^2}{l^6}\,\frac{1}{3}\left(\frac{l}{2}\right)^3$$

Equating

$$\omega_1^2 = 453\,\frac{EI}{(m_0 l)\,l^3} \qquad \omega_1 = 21.30\sqrt{\frac{EI}{(m_0 l)\,l^3}}$$

To check, first find total mass

$$\int m\,dx = 2m_0\int_0^{l/2}\left(\frac{x}{l} - \frac{x^3}{l^3}\right)dx = 2m_0\left[\frac{x^2}{2l} - \frac{x^4}{4l^3}\right]_0^{l/2} = 0.2188\ m_0 l = M_T$$



For massless beam with M at midspan $k = \frac{48EI}{l^3}$

$$T = \frac{1}{2}\,m_{eff}\,\cancel{\omega^2}\,\cancel{y_m^2} = 0.0529\ m_0 l\,\cancel{\omega^2}\,\cancel{y_m^2} \text{ from above}$$

$$\therefore m_{eff} = .1058\ m_0 l \quad = N\ M_T = N\ .2188\ m_0 l$$

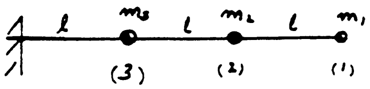
$$\therefore N = .4835 \qquad \therefore m_{eff} = .4835\ (\text{total mass})$$

$$\omega^2 = \frac{k}{m_{eff}} = \frac{48EI}{.4835 M_T\,l^3} = 99.28\ \frac{EI}{M_T l^3}$$

$$\omega = 9.96\sqrt{\frac{EI}{M_T l^3}}$$ $\therefore$ results appear to be reasonable

12-12

From Eq. 12.2-3, $\frac{1}{\omega_1^2} \cong a_{11} m_1 + a_{22} m_2 + a_{33} m_3$



$$a_{11} = \frac{(3l)^3}{3EI} = \frac{27 l^3}{3EI}$$

$$a_{22} = \frac{(2l)^3}{3EI} = \frac{8l^3}{3EI}$$

$$a_{33} = \frac{l^3}{3EI}$$

$$\therefore \frac{1}{\omega_1^2} \cong (27 m_1 + 8 m_2 + m_3)\frac{l^3}{3EI} \qquad \omega_1 \cong \sqrt{\frac{3EI}{l^3}\left(\frac{1}{27m_1 + 8m_2 + m_3}\right)}$$

12-13

Flex. $a_{xb} = \frac{bx}{6EIl}(l^2 - x^2 - b^2)$

$$a_{11} = \frac{\frac{3}{4}l \cdot \frac{1}{4}l}{6EI\,l}\left(l^2 - \frac{l^2}{16} - \frac{9}{16}l^2\right) = \frac{3}{16^2}\frac{l^3}{EI} = a_{33}$$

$$a_{22} = \frac{\frac{1}{2}l \cdot \frac{1}{2}l}{6EIl}\left(l^2 - \frac{l^2}{4} - \frac{l^2}{4}\right) = \frac{1}{48}\frac{l^3}{EI} = \frac{16}{3}\cdot\frac{l^3}{16^2 EI}$$

$$\frac{1}{\omega_1^2} \cong a_{11}\frac{W_1}{g} + a_{22}\frac{W_2}{g} + a_{33}\frac{W_3}{g}$$

$$= \frac{l^3}{16^2 EIg}\left(3W_1 + \frac{16}{3}W_2 + 3W_3\right) = \frac{l^3 W}{16^2 EI}\left(3 + \frac{16}{3}\times 4 + 6\right)$$

$$\therefore \omega_1 \cong 16\sqrt{\frac{3}{91}\frac{EIg}{Wl^3}} = 2.905\sqrt{\frac{EIg}{Wl^3}}$$

12-14

Use Eq. (b) Ex 12.2-1 with $a_{22} = \frac{0.78}{1000}$

$$\frac{1}{\omega_1^2} \cong \frac{1}{\omega_{11}^2} + \frac{0.78}{1000}\frac{320}{386} = \left(\frac{60}{2\pi \times 622}\right)^2 + 0.6466\times 10^{-3}$$

$$= (0.2357 + 0.6466)\times 10^{-3}$$

$$= 0.8823 \times 10^{-3}$$

$$\omega_1 \cong \sqrt{\frac{1}{0.8823\times 10^{-3}}} = 33.66 \text{ rad/sec} = 5.358 \text{ Hz.}$$

12-15

$$\frac{1}{(\omega_1)_1^2} - \frac{1}{\omega_{11}^2} = a_{22}(m_2)_1$$

$$\frac{1}{(\omega_1)_2^2} - \frac{1}{\omega_{11}^2} = a_{22}(m_2)_2 \qquad \text{divide to eliminate } a_{22}$$

$$\frac{\omega_{11}^2 - (\omega_1)_1^2}{(\omega_1)_1^2\,\cancel{\omega_{11}^2}}\;\frac{(\omega_1)_2^2\,\cancel{\omega_{11}^2}}{\omega_{11}^2 - (\omega_1)_2^2} = \frac{(m_2)_1}{(m_2)_2}$$

$$(\omega_1)_1 = 435 \times 2\pi \qquad (m_2)_1 = 5.44$$

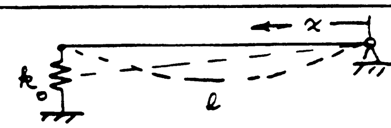
$$(\omega_1)_2 = 398 \times 2\pi \qquad (m_2)_2 = 5.44 + 4.52 = 9.96$$

$$\left(\frac{398}{435}\right)^2 \left[\frac{f_{11}^2 - 435^2}{f_{11}^2 - 398^2}\right] = \frac{5.44}{9.96} \qquad \therefore f_{11}^2 = 245219$$

$$f_{11} = 495.2 \text{ cps}$$

12-16

$$\phi_1 = \frac{x}{l}, \qquad \phi_2 = \sin\frac{\pi x}{l}$$



$$m_{11} = \int m\,\phi_1\phi_1\,dx = \frac{m}{l^2}\int_0^l x^2\,dx = \frac{ml}{3}$$

$$m_{12} = \int_0^l m\,\frac{x}{l}\cdot\sin\frac{\pi x}{l}\,dx = \frac{ml}{\pi^2}\int_0^{\frac{\pi x}{l}=\pi}\left(\frac{\pi x}{l}\right)\sin\left(\frac{\pi x}{l}\right)d\left(\frac{\pi x}{l}\right)$$

$$= \frac{ml}{\pi^2}\left[\sin\frac{\pi x}{l} - \left(\frac{\pi x}{l}\right)\cos\frac{\pi x}{l}\right]_0^{\frac{\pi x}{l}=\pi} = \frac{ml}{\pi}$$

$$m_{22} = m\int_0^l \sin^2\frac{\pi x}{l}\,dx = m\int_0^l \frac{1}{2}\left[1 - \cos\frac{2\pi x}{l}\right]dx = \frac{ml}{2}$$

12-16 Cont.

$$EI\int_0^\ell \phi_1''\phi_1''\,dx = 0 \qquad EI\int_0^\ell \phi_1''\,\phi_2''\,dx = 0$$

$$EI\int_0^\ell \phi_2''\phi_2''\,dx = EI\left(\frac{\pi}{\ell}\right)^4\int_0^\ell \sin^2\frac{\pi x}{\ell}\,dx = \left(\frac{\pi}{\ell}\right)^4 EI\,\frac{\ell}{2}$$

$$u = C_1\,\phi_1(x) + C_2\,\phi_2(x)$$

$$U = \frac{1}{2}\int_0^\ell EI\,(u'')^2\,dx + \frac{1}{2}k_0\,u^2(\ell) \qquad \text{where } u^2(\ell) = C_1^2$$

$$= \frac{1}{2}k_0 C_1^2 + \frac{1}{2}\left(\frac{\pi}{\ell}\right)^4 EI\frac{\ell}{2}\;C_2^2$$

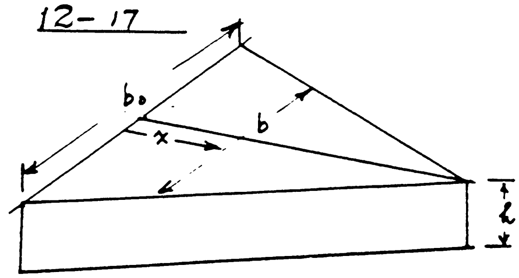
$$\frac{\partial U}{\partial C_1} = k_0 C_1 \qquad \frac{\partial U}{\partial C_2} = \left(\frac{\pi}{\ell}\right)^4 EI\frac{\ell}{2}\;C_2$$

$\therefore$ Eq. 12.3-7 becomes

$$\begin{bmatrix} \left(k_0 - \omega^2\frac{m\ell}{3}\right) & -\omega^2\frac{m\ell}{\pi} \\ -\omega^2\frac{m\ell}{\pi} & \left\{\left(\frac{\pi}{\ell}\right)^4 EI\frac{\ell}{2} - \omega^2\frac{m\ell}{2}\right\} \end{bmatrix}\begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \{0\}$$

$$\left[\left(\frac{1}{6}-\frac{1}{\pi^2}\right)(m\ell)^2\right]\omega^4 - m\ell\left[\frac{\pi^4 EI}{6\,\ell^3} + \frac{k_0}{2}\right]\omega^2 + \frac{k_0}{2}\pi^4\frac{EI}{\ell^3} = 0$$

12-17



$$m(x) = m_0\left(1 - \frac{x}{\ell}\right)$$

$$b(x) = b_0\left(1 - \frac{x}{\ell}\right)$$

$$I(x) = \frac{b h^3}{12} = \frac{h^3}{12} b_0\left(1 - \frac{x}{\ell}\right)$$

$$\therefore\ EI = EI_0\left(1 - \frac{x}{\ell}\right)$$

$$y = C_1 x^2 + C_2 x^3 = C_1 \phi_1 + C_2 \phi_2$$

$$\phi_1 = x^2 \qquad \phi_2 = x^3 \qquad k_{ij} = \int_0^\ell EI\, \phi_i'' \phi_j''\, dx$$

$$\phi_1'' = 2 \qquad \phi_2'' = 6x \qquad m_{ij} = \int m(x)\, \phi_i \phi_j\, dx$$

$$k_{11} = 4EI_0 \int_0^\ell \left(1 - \frac{x}{\ell}\right) dx = 4EI_0\left(\ell - \frac{\ell}{2}\right) = 2EI_0 \ell$$

$$k_{12} = 12EI_0 \int_0^\ell x\left(1 - \frac{x}{\ell}\right) dx = 12EI_0\left(\frac{\ell^2}{2} - \frac{\ell^3}{3\ell}\right) = 2EI_0 \ell^2$$

$$k_{22} = 36EI_0 \int_0^\ell x^2\left(1 - \frac{x}{\ell}\right) dx = 36\left(\frac{\ell^3}{3} - \frac{\ell^4}{4\ell}\right) = 3EI_0 \ell^3$$

$$m_{11} = m_0 \int_0^\ell x^4\left(1 - \frac{x}{\ell}\right) dx = m_0\left(\frac{\ell^5}{5} - \frac{\ell^6}{6\ell}\right) = \frac{1}{30} m_0 \ell^5$$

$$m_{12} = m_0 \int_0^\ell x^5\left(1 - \frac{x}{\ell}\right) dx = m_0\left(\frac{\ell^6}{6} - \frac{\ell^6}{7}\right) = \frac{1}{42} m_0 \ell^6$$

$$m_{22} = m_0 \int_0^\ell x^6\left(1 - \frac{x}{\ell}\right) dx = m_0\left(\frac{\ell^7}{7} - \frac{\ell^7}{8}\right) = \frac{1}{56} m_0 \ell^7$$

12-17 Cont. Subst. into Eq. 12.3-7

$$\begin{bmatrix} \left(2EI_0\ell - \omega^2 \frac{m\ell^5}{30}\right) & \left(2EI_0\ell^2 - \omega^2 \frac{m_0\ell^6}{42}\right) \\ \left(2EI_0\ell^2 - \omega^2 \frac{m_0\ell^6}{42}\right) & \left(3EI_0\ell^3 - \omega^2 \frac{m_0\ell^7}{56}\right) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \{0\}$$

Freq. Eq. from determinant of above eq $= 0$

$$6(EI_0)^2\ell^4 - \omega^2\left[\frac{m_0\ell^5}{30}3EI_0\ell^3 + \frac{m_0\ell^7}{56}2EI_0\ell\right] + \omega^4\left(\frac{m_0\ell^5}{30}\cdot\frac{m_0\ell^7}{56}\right)$$

$$-4(EI_0)^2\ell^4 + \left[4EI_0\ell^2\frac{m\ell^6}{42}\right]\omega^2 - \omega^4\left(\frac{m_0^2\ell^{12}}{42^2}\right) = 0$$

$$\omega^4\left[m_0^2\ell^{12}\left(\frac{1}{30\times 56} - \frac{1}{42^2}\right)\right] - \omega^2\left[m_0EI_0\ell^8\left(\frac{3}{30} + \frac{2}{56} - \frac{4}{42}\right)\right] + 2(EI_0)^2\ell^4 = 0$$

$$\left(28.345\times 10^{-6}\, m_0^2\ell^{12}\right)\omega^4 - \left(0.0405\, m_0 EI_0\ell^3\right)\omega^2 + 2(EI_0)^2\ell^4 = 0$$

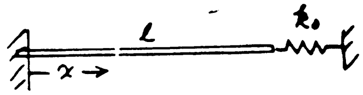
$$\omega^4 - 1429.3\left(\frac{EI_0}{m_0\ell^4}\right)\omega^2 + 70559.\left(\frac{EI_0}{m_0\ell^4}\right)^2 = 0$$

$$\omega^2 = \begin{Bmatrix} 51.20 \\ 1378 \end{Bmatrix} \times \frac{EI_0}{m_0\ell^4}$$

$$\omega_1 = 7.155\sqrt{\frac{EI_0}{m_0\ell^4}}$$

$$\omega_2 = 37.12\sqrt{\frac{EI_0}{m_0\ell^4}}$$

12-18



Normal modes of fixed-free uniform rods are

$$\phi_1 = \sin\frac{\pi}{2}\frac{x}{\ell} \qquad \phi_2 = \sin\frac{3\pi}{2}\frac{x}{\ell} \qquad \text{See Eq. 9.2-8}$$

$$\therefore \quad u(x) = C_1 \sin\frac{\pi}{2}\frac{x}{\ell} + C_2 \sin\frac{3\pi}{2}\frac{x}{\ell}$$

$$\frac{\partial u}{\partial x} = C_1 \frac{\pi}{2\ell}\cos\frac{\pi}{2}\frac{x}{\ell} + C_2\frac{3\pi}{2}\cos\frac{3\pi}{2}\frac{x}{\ell}$$

$$U = \frac{1}{2}AE\int_0^\ell \left(\frac{\partial u}{\partial x}\right)^2 dx + \frac{1}{2}k_o u^2(\ell)$$

$$= \frac{1}{2}AE\int_0^\ell C_1^2\left(\frac{\pi}{2\ell}\right)^2\cos^2\frac{\pi}{2}\frac{x}{\ell}\,dx + AE\,C_1C_2\int_0^\ell\left(\frac{\pi}{2\ell}\right)\left(\frac{3\pi}{2\ell}\right)\cos\frac{\pi}{2}\frac{x}{\ell}\cos\frac{3\pi}{2}\frac{x}{\ell}\,dx$$

$$+ \frac{1}{2}AE\int_0^\ell C_2^2\left(\frac{3\pi}{2\ell}\right)^2\cos^2\frac{3\pi}{2\ell}x\,dx + \frac{1}{2}k_o(C_1 - C_2)^2$$

$$\frac{\partial U}{\partial C_1} = \left[AE\left(\frac{\pi}{2\ell}\right)^2\int_0^\ell \cos^2\frac{\pi}{2}\frac{x}{\ell}\,dx + k_o\right]C_1 - k_o C_2$$

$$+ AE\,C_2\int_0^\ell \cancel{\left(\frac{\pi}{2\ell}\right)\left(\frac{3\pi}{2\ell}\right)\cos\frac{\pi}{2}\frac{x}{\ell}\cos\frac{3\pi}{2}\frac{x}{\ell}\,dx} \to 0$$

$$\frac{\partial U}{\partial C_2} = \left[AE\left(\frac{3\pi}{2\ell}\right)^2\int_0^\ell \cos^2\frac{3\pi}{2\ell}x\,dx\right]C_2 - k_oC_1 + k_oC_2$$

$$\therefore \quad k_{11} = AE\left(\frac{\pi}{2\ell}\right)^2\int_0^\ell \cos^2\frac{\pi}{2}\frac{x}{\ell}\,dx + k_o$$

$$k_{12} = k_{21} = -k_o$$

$$k_{22} = AE\left(\frac{3\pi}{2\ell}\right)^2\int_0^\ell \cos^2\frac{3\pi}{2}\frac{x}{\ell}\,dx + k_o$$

12-18 Cont:

Since $\displaystyle\int_0^{\ell} \cos^2 n\theta\, d\theta = \frac{\ell}{2} \qquad n = \frac{\pi}{2\ell},\ \frac{3\pi}{2\ell},$

$$k_{11} = AE\left(\frac{\pi}{2\ell}\right)^2\frac{\ell}{2} + k_0 \qquad k_{12} = -k_0$$

$$k_{22} = AE\left(\frac{3\pi}{2\ell}\right)^2\frac{\ell}{2} + k_0$$

$$m_{ij} = \int_0^{\ell} m\,\phi_i\phi_j\,dx = \begin{cases} m_{11} = \dfrac{m\ell}{2} \\ m_{12} = 0 \\ m_{22} = \dfrac{m\ell}{2} \end{cases}$$

Subst into Eq. 12.3-7

$$\begin{bmatrix} \left\{AE\left(\frac{\pi}{2\ell}\right)^2\frac{\ell}{2} + k_0 - \omega^2\frac{m\ell}{2}\right\} & -k_0 \\ -k_0 & \left\{AE\left(\frac{3\pi}{2\ell}\right)^2\frac{\ell}{2} + k_0 - \omega^2\frac{m\ell}{2}\right\} \end{bmatrix}\begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \{0\}$$

Freq. Eq.

$$\left(\frac{m\ell}{2}\omega^2\right)^2 - \left(\frac{m\ell}{2}\omega^2\right)\left[AE\left(\frac{\pi}{2\ell}\right)^2\frac{\ell}{2} + k_0 + AE\left(\frac{3\pi}{2\ell}\right)^2\frac{\ell}{2} + k_0\right]$$
$$+ \left[AE\left(\frac{\pi}{2\ell}\right)^2\frac{\ell}{2} + k_0\right]\left[AE\left(\frac{3\pi}{2\ell}\right)^2\frac{\ell}{2} + k_0\right] - k_0^2 = 0$$

Reduces to

$$\omega^4 - \omega^2\left[10\left(\frac{\pi}{2\ell}\right)^2\frac{AE}{m} + \frac{4k_0^2}{m\ell}\right] + \left[9\left(\frac{\pi}{2\ell}\right)^4\left(\frac{AE}{m}\right)^2 + 20\left(\frac{\pi}{2\ell}\right)^2\frac{AE}{m}\frac{k_0}{m\ell}\right] = 0$$

12-19 Refer to Prob 12-18

With $k_o = 0$

$$k_{11} = AE\left(\frac{\pi}{2\ell}\right)^2 \frac{\ell}{2} \qquad k_{12} = 0$$

$$k_{22} = AE\left(\frac{3\pi}{2\ell}\right)^2 \frac{\ell}{2}$$

$$\text{Additional } KE = \frac{1}{2} m_o \dot{u}^2_{(\ell)} = \frac{1}{2} m_o (\dot{c}_1 - \dot{c}_2)^2$$

$$= \frac{1}{2} m_o (\dot{c}_1^2 - 2\dot{c}_1\dot{c}_2 + \dot{c}_2^2)$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{c}_1} = m_o \ddot{c}_1 - m_o \ddot{c}_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{c}_2}\right) = -m_o \ddot{c}_1 + m_o \ddot{c}_2$$

$$\therefore \; m_{11} = m\frac{\ell}{2} + m_o$$

$$m_{12} = m_{21} = -m_o$$

$$m_{22} = \frac{m\ell}{2} + m_o$$

Eq. 12.3-7 becomes

$$\begin{bmatrix} \left\{AE\left(\frac{\pi}{2\ell}\right)^2\frac{\ell}{2} - \omega^2\left(\frac{m\ell}{2}+m_o\right)\right\} & \omega^2 m_o \\ \omega^2 m_o & \left\{AE\left(\frac{3\pi}{2\ell}\right)^2\frac{\ell}{2} - \omega^2\left(\frac{m\ell}{2}+m_o\right)\right\} \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \{C\}$$

12-20

$$m(x) = m_0 \frac{x}{\ell}\left(1 - \frac{x}{\ell}\right)$$

$$\phi_1 = \sin \frac{\pi x}{\ell} \qquad \phi_2 = \sin \frac{2\pi x}{\ell}$$

$$y = \phi_1 q_1 + \phi_2 q_2$$

$$k_{11} = EI \int_0^\ell \phi_1'' \phi_1'' dx = EI\left(\frac{\pi}{\ell}\right)^4 \int_0^\ell \sin^2 \frac{\pi x}{\ell} dx = EI\left(\frac{\pi}{\ell}\right)^4 \frac{\ell}{2}$$

$$k_{12} = k_{21} = EI \int_0^\ell \phi_1'' \phi_2'' = 0$$

$$k_{22} = EI \int_0^\ell \phi_2'' \phi_2'' dx = EI\left(\frac{2\pi}{\ell}\right)^4 \int_0^\ell \sin^2 \frac{2\pi x}{\ell} dx = EI\left(\frac{2\pi}{\ell}\right)^4 \frac{\ell}{2}$$

$$m_{11} = \int_0^\ell m(x)\phi_1\phi_1 dx = m_0 \int_0^\ell \frac{x}{\ell}\left(1 - \frac{x}{\ell}\right) \sin^2 \frac{\pi x}{\ell} dx$$

$$= m_0 \int_0^\ell \frac{x}{\ell} \frac{1}{2}\left(1 - \cos \frac{2\pi x}{\ell}\right) dx - m_0 \int_0^\ell \left(\frac{x}{\ell}\right)^2 \frac{1}{2}\left(1 - \cos \frac{2\pi x}{\ell}\right) dx$$

$$= m_0 \left[\frac{\ell}{4} - \frac{\ell}{2\pi^2}\left(\frac{\cos\left(\frac{2\pi x}{\ell}\right)}{\left(\frac{2\pi}{\ell}\right)^2} - \left(\frac{\pi x}{\ell}\right) \frac{\sin\left(\frac{2\pi x}{\ell}\right)}{\left(\frac{2\pi}{\ell}\right)} \right)\Bigg|_0^\ell \right]$$

$$- m_0 \left[\frac{\ell}{6} - \frac{\ell}{16\pi^3} \int_0^\ell \left(\frac{2\pi x}{\ell}\right)^2 \cos \frac{2\pi x}{\ell} \, d\left(\frac{2\pi x}{\ell}\right) \right]$$

$$= \frac{m_0 \ell}{12} + \frac{m_0 \ell}{16\pi^3} \left[2\left(\frac{2\pi x}{\ell}\right) \cos \frac{2\pi x}{\ell} + \left\{ \left(\frac{2\pi x}{\ell}\right)^2 - 2 \right\} \sin \frac{2\pi x}{\ell} \right]_0^\ell$$

$$= \frac{m_0 \ell}{12} + \frac{m_0 \ell}{4\pi^2} = 0.10866\, m_0 \ell$$

12-20 Cont:

$$m_{22} = m_o\int_0^l \left(\frac{x}{l}\right)\left(1-\frac{x}{l}\right)\sin^2\frac{2\pi x}{l}\,dx = m_o\int_0^l \frac{x}{l}\left(1-\frac{x}{l}\right)\frac{1}{2}\left(1-\cos\frac{4\pi x}{l}\right)dx$$

$$= \frac{m_o l}{12} - \frac{m_o}{2}\int_0^l \frac{x}{l}\left(1-\frac{x}{l}\right)\cos\frac{4\pi x}{l}\,dx = \frac{m_o l}{12} + \frac{m_o l}{(4\pi)^2}$$

$$= 0.08966\ m_o l$$

$$m_{12} = m_o\int_0^l \left(\frac{x}{l}\right)\left(1-\frac{x}{l}\right)\sin\frac{\pi x}{l}\sin\frac{2\pi x}{l}\,dx = 0 \quad \text{by inspection}$$

ie $\frac{x}{l}\left(1-\frac{x}{l}\right)$ = symmetric function

$\sin\frac{\pi x}{l}\sin\frac{2\pi x}{l}$ = unsymmetric function

Eq. (12.3-7) becomes

$$\begin{bmatrix} \left[EI\left(\frac{\pi}{l}\right)^4\frac{l}{2} - 0.10866\,m_o l\,\omega^2\right] & [\quad 0 \quad] \\ [\quad 0 \quad] & \left[EI\left(\frac{2\pi}{l}\right)^4\frac{l}{2} - 0.08966\,m_o l\,\omega^2\right] \end{bmatrix}\begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = 0$$

$\therefore$ C_1 & C_2 are independent & Rayleigh-Ritz method fails.

However from above

$$\left[EI\left(\frac{\pi}{l}\right)^4\frac{l}{2} - 0.10866\,m_o l\,\omega^2\right] = 0 \qquad \left[EI\left(\frac{2\pi}{l}\right)^4\frac{l}{2} - 0.08966\,m_o l\,\omega^2\right] = 0$$

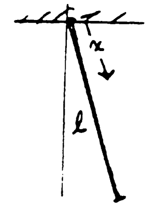
reduces to two Rayleigh's method.

$$\omega_1^2 = \frac{\pi^4}{2\times .10866}\left(\frac{EI}{m_o l^4}\right) = 448\frac{EI}{m_o l^4} \qquad \omega_1 = 21.2\sqrt{\frac{EI}{m_o l^4}}$$

$$\omega_2^2 = \frac{8\pi^4}{.08966}\left(\frac{EI}{m_o l^4}\right) = 8691\frac{EI}{m_o l^4} \qquad \omega_2 = 93\sqrt{\frac{EI}{m_o l^4}}$$

1st mode is sine $\frac{\pi x}{l}$, 2nd mode is $\sin\frac{2\pi x}{l}$. Compare with Prob 12-11. (ω_1 agrees)

12-21



$$\phi_1 = \frac{x}{\ell} \qquad \phi_1'' = 0$$

$$\phi_2 = \sin\frac{\pi x}{\ell} \qquad \phi_2'' = -\left(\frac{\pi}{\ell}\right)^2 \sin\frac{\pi x}{\ell}$$

$$\phi_3 = \sin\frac{2\pi x}{\ell} \qquad \phi_3'' = -\left(\frac{2\pi}{\ell}\right)^2 \sin\frac{2\pi x}{\ell}$$

$$k_{11} = EI\int_0^\ell \phi_1''\phi_1''\,dx = 0 \qquad k_{12} = k_{13} = 0$$

$$k_{22} = EI\left(\frac{\pi}{\ell}\right)^4\int_0^\ell \sin^2\frac{\pi x}{\ell}\,dx = EI\left(\frac{\pi}{\ell}\right)^4\frac{\ell}{2}$$

$$k_{33} = EI\left(\frac{2\pi}{\ell}\right)^4\frac{\ell}{2}$$

$$k_{23} = EI\left(\frac{\pi}{\ell}\right)^2\left(\frac{2\pi}{\ell}\right)^2\int_0^\ell \sin\frac{\pi x}{\ell}\cdot\sin\frac{2\pi x}{\ell}\,dx = 0$$

$$m_{11} = m_0\int_0^\ell \phi_1^2\,dx = m_0\int_0^\ell\left(\frac{x}{\ell}\right)^2 dx = \frac{m_0\ell}{3}$$

$$m_{22} = m_0\int_0^\ell \sin^2\frac{\pi x}{\ell}\,dx = \frac{m_0\ell}{2}$$

$$m_{33} = m_0\int_0^\ell \sin^2\frac{2\pi x}{\ell}\,dx = \frac{m_0\ell}{2}$$

$$m_{12} = m_0\int_0^\ell \frac{x}{\ell}\sin\frac{\pi x}{\ell}\,dx = \frac{m_0}{\ell}\left[\frac{\sin\frac{\pi x}{\ell}}{\left(\frac{\pi}{\ell}\right)^2} - x\frac{\cos\frac{\pi x}{\ell}}{\left(\frac{\pi}{\ell}\right)}\right]_0^\ell$$

$$= \frac{m_0\ell}{\pi} = m_{21}$$

$$m_{13} = m_0\int_0^\ell \frac{x}{\ell}\sin\frac{2\pi x}{\ell}\,dx = -\frac{m_0\ell}{2\pi}$$

12-21 Cont

$$m_{23} = m_0 \int_0^{\ell} \sin\frac{\pi x}{\ell} \sin\frac{2\pi x}{\ell}\, dx = 0$$

$$\begin{bmatrix} \left(0 - \omega^2 \frac{m_0\ell}{3}\right) & \left(0 - \omega^2 \frac{m_0\ell}{\pi}\right) & \left(0 + \omega^2 \frac{m_0\ell}{2\pi}\right) \\ \left(0 - \omega^2 \frac{m_0\ell}{\pi}\right) & \left(EI\left(\frac{\pi}{\ell}\right)^4 \frac{\ell}{2} - \omega^2 \frac{m_0\ell}{2}\right) & (0 - 0) \\ \left(0 - \omega^2 \frac{m_0\ell}{2\pi}\right) & (0 - 0) & \left(EI\left(\frac{2\pi}{\ell}\right)^4 \frac{\ell}{2} - \omega^2 \frac{m_0\ell}{2}\right) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix} = \{0\}$$

Freq. Eq.

$$-\omega^2 \frac{m_0\ell}{3}\left[EI\left(\frac{\pi}{\ell}\right)^4 \frac{\ell}{2} - \omega^2 \frac{m_0\ell}{2}\right]\left[EI\left(\frac{2\pi}{\ell}\right)^4 \frac{\ell}{2} - \omega^2 \frac{m_0\ell}{2}\right]$$

$$+ \omega^2 \frac{m_0\ell}{\pi}\left[-\left(\omega^2 \frac{m_0\ell}{\pi}\right)\left\{EI\left(\frac{2\pi}{\ell}\right)^4 \frac{\ell}{2} - \omega^2 \frac{m_0\ell}{2}\right\}\right]$$

$$- \omega^2 \frac{m_0\ell}{2\pi}\left[-\left(\omega^2 \frac{m_0\ell}{2\pi}\right)\left\{EI\left(\frac{\pi}{\ell}\right)^4 \frac{\ell}{2} - \omega^2 \frac{m_0\ell}{2}\right\}\right] = 0$$

$$-(\omega^2 m_0\ell)^3\left[\frac{1}{12} - \frac{1}{2\pi^2} + \frac{1}{8\pi^2}\right] + (\omega^2 m_0\ell)^2 EI\left[\frac{\ell}{12}\left(\frac{\pi}{\ell}\right)^4 + \frac{\ell}{12}\left(\frac{2\pi}{\ell}\right)^4 - \frac{\ell}{2\pi^2}\left(\frac{2\pi}{\ell}\right)^4 + \frac{\ell}{8\pi^2}\left(\frac{\pi}{\ell}\right)^4\right]$$

$$-(\omega^2 m_0\ell)(EI)^2 \frac{\ell^2}{12}\left(\frac{\pi}{\ell}\right)^4\left(\frac{2\pi}{\ell}\right)^4 = 0$$

$$-0.045338(\omega^2)^3 + 60.273(\omega^2)^2\left(\frac{EI}{m_0\ell^4}\right) - 12651.37\,\omega^2\left(\frac{EI}{m_0\ell^4}\right)^2 = 0$$

$$(\omega^2)^2 - \left(1329.4146 \frac{EI}{m_0\ell^4}\right)\omega^2 + 279045.61\left(\frac{EI}{m_0\ell^4}\right)^2 = 0$$

$$\omega^2 = (664.70 \pm 403.47)\left(\frac{EI}{m_0\ell^4}\right) = \begin{Bmatrix} 261.23 \\ 1068.17 \end{Bmatrix} \frac{EI}{m_0\ell^4}$$

$$\omega = \begin{Bmatrix} 0.0 \\ 16.16 \\ 32.68 \end{Bmatrix} \sqrt{\frac{EI}{m_0\ell^4}} \qquad \text{Exact Sol} \quad \omega = \begin{Bmatrix} 0 \\ 15.4 \\ 50. \end{Bmatrix} \sqrt{\frac{EI}{m_0\ell^4}}$$

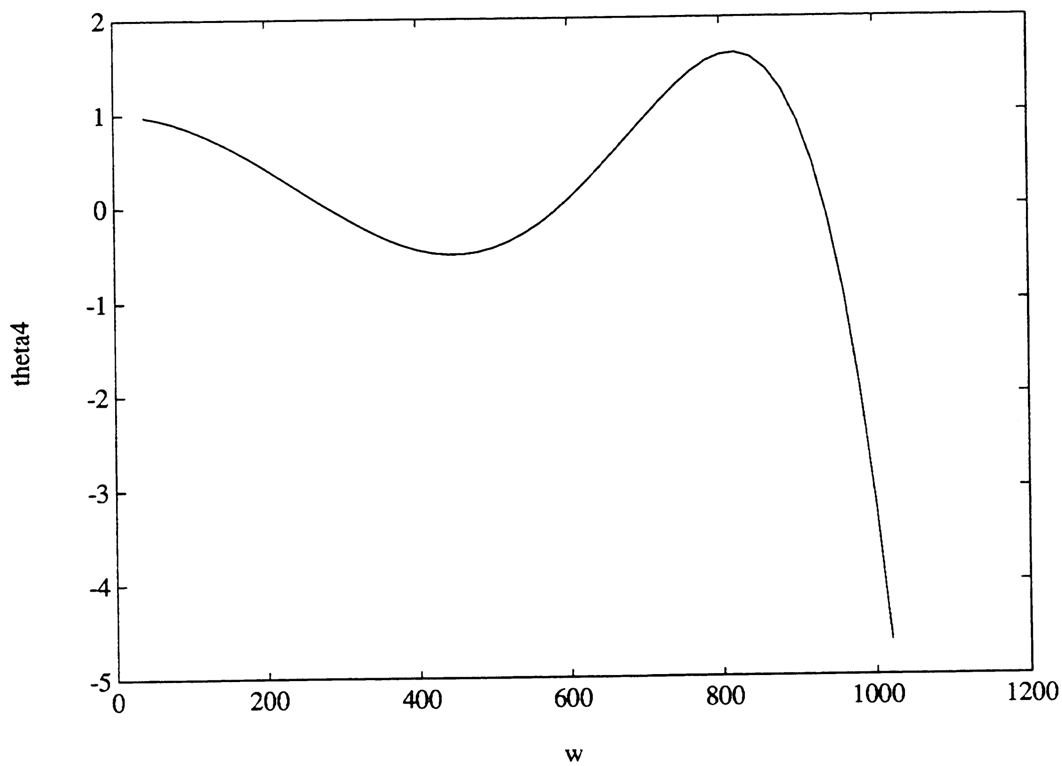
12-22

use the program tor.m

$J = [1, 1, 2, 0]$

$K = [.2, .4, .6] \times 10^6$

$I = 1:50$



12-22 cont

```
»[w',theta4']

ans =

   1.0e+03 *

    0.0400    0.0010
    0.0600    0.0009
    0.0800    0.0009
    0.1000    0.0008
    0.1200    0.0008
    0.1400    0.0007
    0.1600    0.0006
    0.1800    0.0005
    0.2000    0.0004
    0.2200    0.0003
    0.2400    0.0002
    0.2600    0.0001
    0.2800    0.0000  ← 1st mode
    0.3000   -0.0001
    0.3200   -0.0002
    0.3400   -0.0003
    0.3600   -0.0003
    0.3800   -0.0004
    0.4000   -0.0005
    0.4200   -0.0005
    0.4400   -0.0005
    0.4600   -0.0005
    0.4800   -0.0005
    0.5000   -0.0004
    0.5200   -0.0004
    0.5400   -0.0003
    0.5600   -0.0002
    0.5800   -0.0001
    0.6000    0.0001  ← 2nd mode
    0.6200    0.0002
    0.6400    0.0004
    0.6600    0.0006
    0.6800    0.0008
    0.7000    0.0009
    0.7200    0.0011
    0.7400    0.0013
    0.7600    0.0014
    0.7800    0.0015
    0.8000    0.0016
```

```
   w'        theta4

1.0e+03*

    0.8200    0.0016
    0.8400    0.0016
    0.8600    0.0014
    0.8800    0.0012
    0.9000    0.0009
    0.9200    0.0004
    0.9400   -0.0002  ← 3rd mode
    0.9600   -0.0010
    0.9800   -0.0020
    1.0000   -0.0032
    1.0200   -0.0046
```

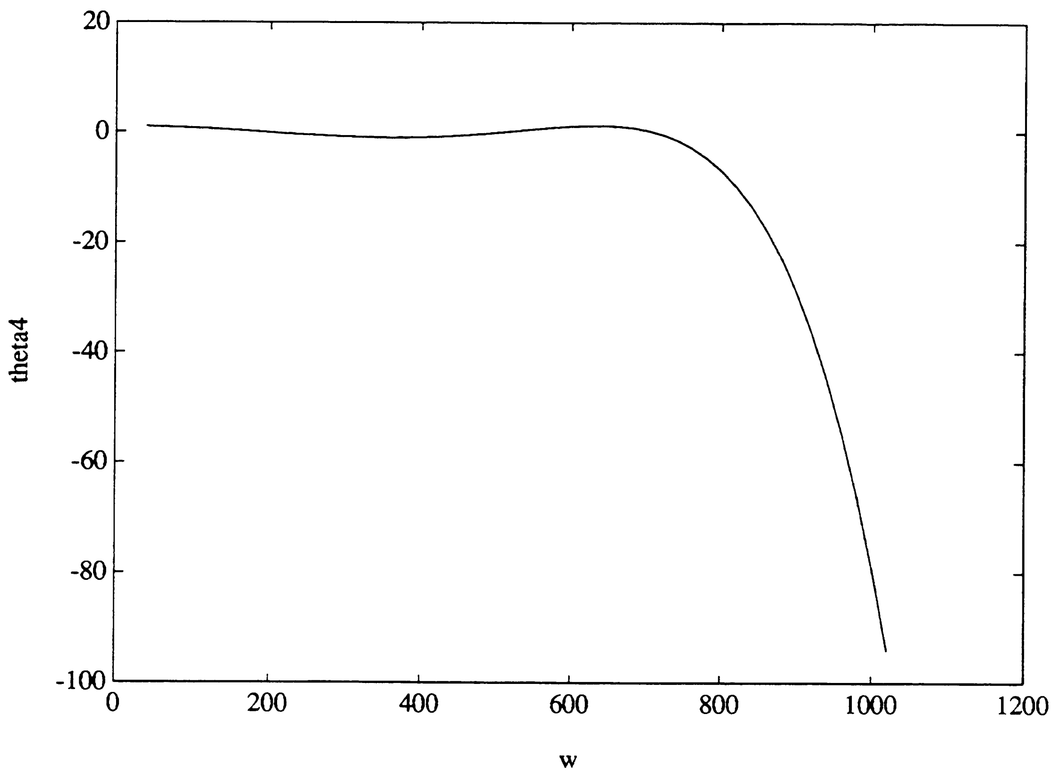
12 - 23

Use Tor.m

$J = [1.13, 1.13, 1.13, 2.26]$

$k = [.169, 1.69, .224] \times 10^6$

$I = 1:50$.



12-25 cont.

```
>>>[w',theta4']

ans =

   1.0e+03 *
```

0.0400	0.0009
0.0600	0.0009
0.0800	0.0008
0.1000	0.0007
0.1200	0.0005
0.1400	0.0004
0.1600	0.0002
0.1800	0.0000
0.2000	-0.0001 ← 1st mode
0.2200	-0.0003
0.2400	-0.0005
0.2600	-0.0006
0.2800	-0.0008
0.3000	-0.0009
0.3200	-0.0010
0.3400	-0.0010
0.3600	-0.0010
0.3800	-0.0010
0.4000	-0.0010
0.4200	-0.0009
0.4400	-0.0007
0.4600	-0.0005
0.4800	-0.0003
0.5000	-0.0001
0.5200	0.0001 ← 2nd mode
0.5400	0.0004
0.5600	0.0007
0.5800	0.0009
0.6000	0.0011
0.6200	0.0012
0.6400	0.0012
0.6600	0.0011
0.6800	0.0008
0.7000	0.0004
0.7200	-0.0004 ← 3rd mode
0.7400	-0.0014
0.7600	-0.0028
0.7800	-0.0046
0.8000	-0.0070

w' theta4'

1.0 E+03 *

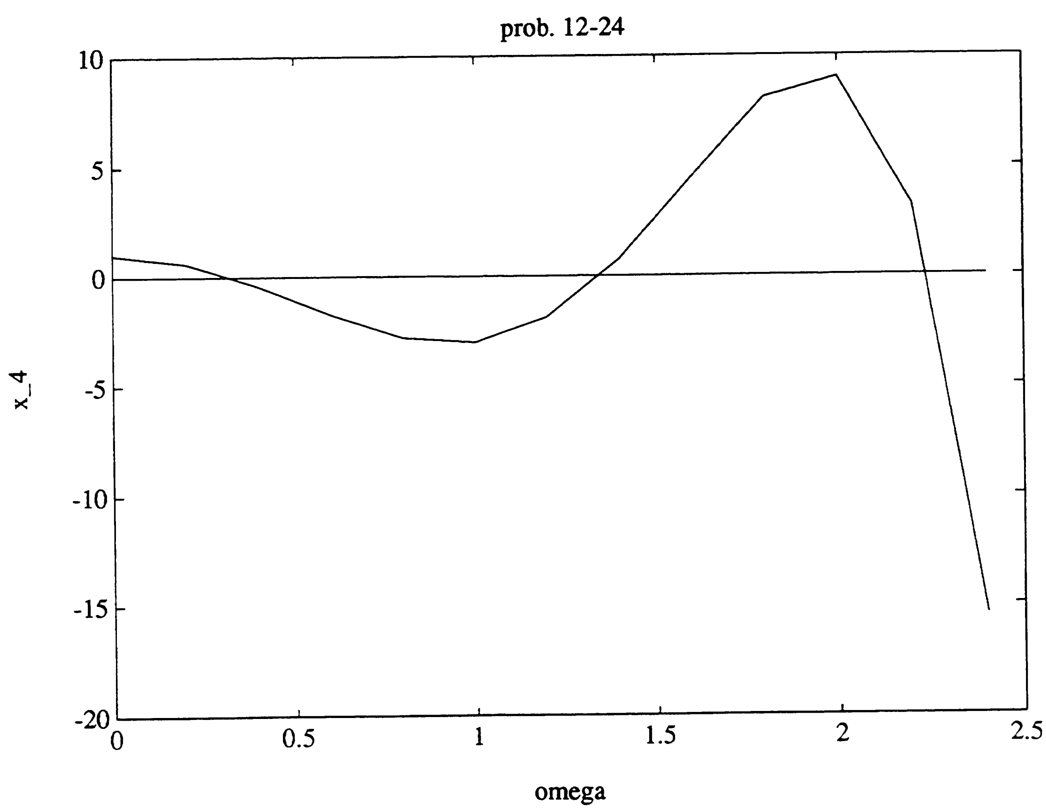
0.8200	-0.0099
0.8400	-0.0134
0.8600	-0.0178
0.8800	-0.0230
0.9000	-0.0291
0.9200	-0.0364
0.9400	-0.0449
0.9600	-0.0547
0.9800	-0.0661
1.0000	-0.0792
1.0200	-0.0941

12-24

```
%This is the Holzer method for the sping mass sysytem
clear;clg;
m=[1 1 1];
k=[1 1 1 ];
for I=1:13
x(1)=1;
 w(I)=(I-1)*.2;
 lambda(I)=(w(I))^2;
  x(2)=1- lambda(I)*m(1)/k(1);
f(1)=lambda(I)*m(1)*x(1);
ff=f(1);
 term=1;
  for ii=2:3
   term=term+x(ii);
   f(ii)=lambda(I)*term;
   ff=ff+f(ii);
   x(ii+1)=x(ii)-ff/k(ii);
end
x4(I)=x(4);
end
plot(w,x4)
```

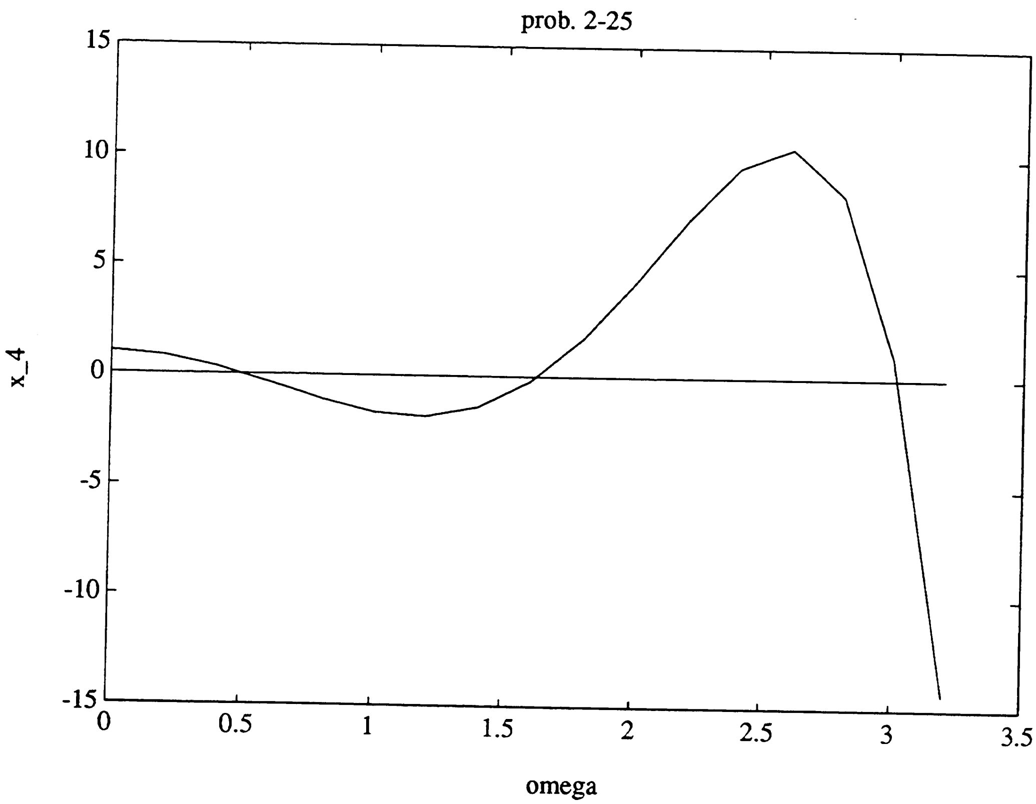
```
[w',x4']

ans =

        0    1.0000
   0.2000    0.6111
   0.4000   -0.4249
   0.6000   -1.7395
   0.8000   -2.7949
   1.0000   -3.0000
   1.2000   -1.8708
   1.4000    0.7617
   1.6000    4.4980
   1.8000    8.0710
   2.0000    9.0000
   2.2000    3.1993
   2.4000  -15.4598
```

$\leftarrow \omega_1$ (at -0.4249), $\leftarrow \omega_2$ (at 0.7617), $\leftarrow \omega_3$ (at 3.1993)



12-25

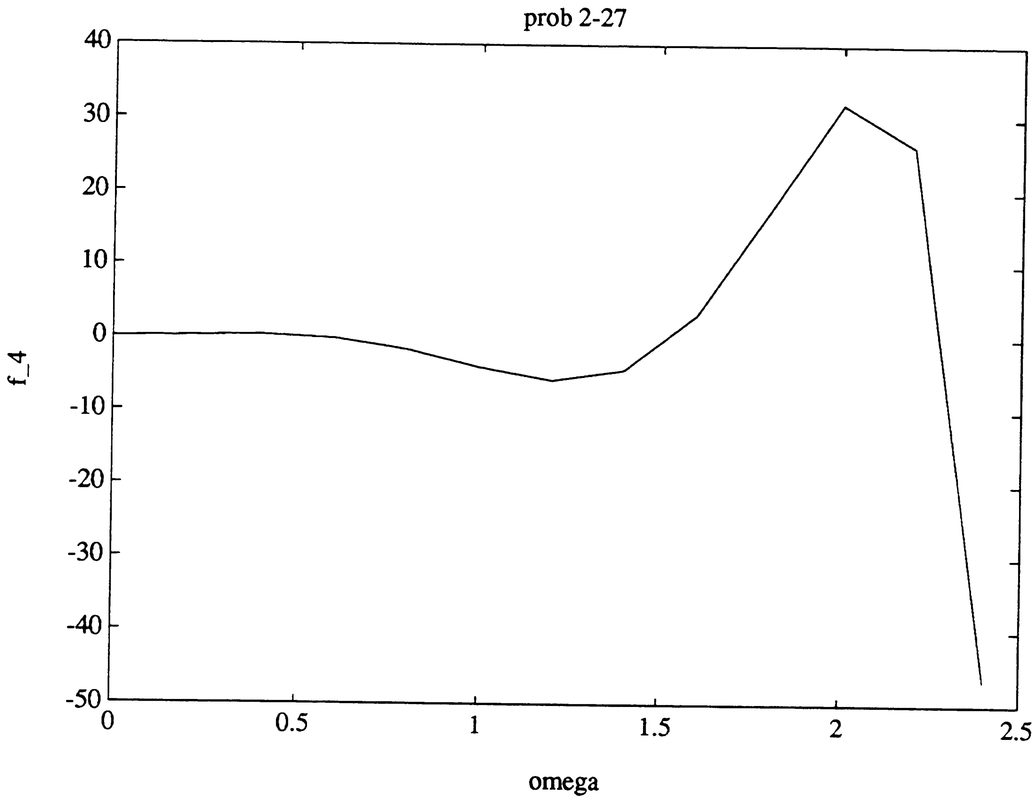
Use the same program as 12-24




```
[w',x4']

ans =

         0    1.0000
    0.2000    0.8232
    0.4000    0.3305 ← ω1
    0.6000   -0.3686
    0.8000   -1.1045
    1.0000   -1.6667
    1.2000   -1.8305
    1.4000   -1.3917
    1.6000   -0.2090 ← ω2
    1.8000    1.7465
    2.0000    4.3333
    2.2000    7.1745
    2.4000    9.5847
    2.6000   10.4892
    2.8000    8.3361
    3.0000    1.0000 ← ω3
    3.2000  -14.3218
```

12-27

Use the same program as 12-25
extend ii index to 4.




```
[w' f4']

ans =
```

0	0	
0.2000	0.1365	
0.4000	0.2881	$\leftarrow \omega_1$
0.6000	-0.1475	
0.8000	-1.6546	
1.0000	-4.0000	
1.2000	-5.7559	
1.4000	-4.3056	$\leftarrow \omega_2$
1.6000	3.2041	
1.8000	17.3942	
2.0000	32.0000	
2.2000	26.2565	$\leftarrow \omega_3$
2.4000	-46.5533	

12-28 Use the program tor.m:

$J = [50, 138, 145, 181, 240, 1/2(14000)]$

$K = 10^6 * [15, 30, 22, 36, 120, 0]$

The system was cut at centerline using 1/2 the J of the fuselage. Antisymmetric modes are given for $\Theta_6 = 0$ and symmetric modes for $T_6 = 0$. Note: they are very close to each other because of the very large J_6

```
»[w' theta6' t6']

ans =

   1.0e+10 *

                   0    0.00000000010000                   0
    0.00000000100000    0.00000000009955    0.00070454393355
    0.00000000200000    0.00000000009819    0.00277996177187
    0.00000000300000    0.00000000009594    0.00611264965261
    0.00000000400000    0.00000000009283    0.01051670778083
    0.00000000500000    0.00000000008889    0.01573903028613
    0.00000000600000    0.00000000008416    0.02146629176526
    0.00000000700000    0.00000000007870    0.02733369347504
    0.00000000800000    0.00000000007256    0.03293529614420
    0.00000000900000    0.00000000006581    0.03783573271471
    0.00000001000000    0.00000000005852    0.04158306348715
    0.00000001100000    0.00000000005078    0.04372250859723
    0.00000001200000    0.00000000004267    0.04381076893278
    0.00000001300000    0.00000000003428    0.04143062692522
    0.00000001400000    0.00000000002570    0.03620550350001
    0.00000001500000    0.00000000001702    0.02781363719313
    0.00000001600000    0.00000000000834    0.01600154634395
    0.00000001700000   -0.00000000000023    0.00059643562529
    0.00000001800000   -0.00000000000862   -0.01848278581237
    0.00000001900000   -0.00000000001672   -0.04121619545193
    0.00000002000000   -0.00000000002446   -0.06747556083990
    0.00000002100000   -0.00000000003173   -0.09701933458121
    0.00000002200000   -0.00000000003846   -0.12949019031575
    0.00000002300000   -0.00000000004459   -0.16441531332092
    0.00000002400000   -0.00000000005004   -0.20120961780241
```

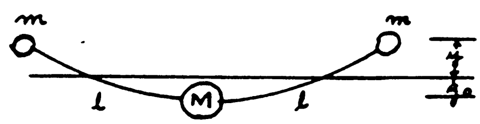
12-28 cont.

ω	θ_6	T_6
0.00000002500000	-0.00000000005475	-0.23918201574982
0.00000002600000	-0.00000000005869	-0.27754480972827
0.00000002700000	-0.00000000006180	-0.31542622451125
0.00000002800000	-0.00000000006406	-0.35188603048073
0.00000002900000	-0.00000000006546	-0.38593414575746
0.00000003000000	-0.00000000006598	-0.41655203470013
0.00000003100000	-0.00000000006564	-0.44271664844260
0.00000003200000	-0.00000000006445	-0.46342657933742
0.00000003300000	-0.00000000006244	-0.47773002645629
0.00000003400000	-0.00000000005966	-0.48475409468513
0.00000003500000	-0.00000000005615	-0.48373487657210
0.00000003600000	-0.00000000005197	-0.47404769518270
0.00000003700000	-0.00000000004721	-0.45523681914657
0.00000003800000	-0.00000000004195	-0.42704389932224
0.00000003900000	-0.00000000003628	-0.38943432166274
0.00000004000000	-0.00000000003030	-0.34262062466385
0.00000004100000	-0.00000000002411	-0.28708209408002
0.00000004200000	-0.00000000001783	-0.22357962439411
0.00000004300000	-0.00000000001157	-0.15316492796073
0.00000004400000	-0.00000000000545	-0.07718318108356
0.00000004500000	0.00000000000042	0.00273177604275
0.00000004600000	0.00000000000592	0.08467051902641
0.00000004700000	0.00000000001094	0.16647015791455
0.00000004800000	0.00000000001539	0.24574615238018
0.00000004900000	0.00000000001916	0.31993434882400
0.00000005000000	0.00000000002218	0.38634359875886
0.00000005100000	0.00000000002436	0.44221911472193
0.00000005200000	0.00000000002566	0.48481647616685
0.00000005300000	0.00000000002603	0.51148590960207
0.00000005400000	0.00000000002546	0.51976613076509
0.00000005500000	0.00000000002394	0.50748664777117
0.00000005600000	0.00000000002149	0.47287697868056
0.00000005700000	0.00000000001817	0.41468073033284
0.00000005800000	0.00000000001403	0.33227191294532
0.00000005900000	0.00000000000918	0.22577022201726
0.00000006000000	0.00000000000372	0.09615130046836
0.00000006100000	-0.00000000000219	-0.05465280558446
0.00000006200000	-0.00000000000840	-0.22366866991186
0.00000006300000	-0.00000000001472	-0.40681863857555
0.00000006400000	-0.00000000002095	-0.59888783801372
0.00000006500000	-0.00000000002688	-0.79352994513813
0.00000006600000	-0.00000000003227	-0.98332191613568
0.00000006700000	-0.00000000003690	-1.15987927799034
0.00000006800000	-0.00000000004056	-1.31404511781761
0.00000006900000	-0.00000000004302	-1.43616756696207

12-28 Cont.

ω	θ_6	T_6
0.00000007000000	-0.00000000004411	-1.51648237671449
0.00000007100000	-0.00000000004366	-1.54561912796732
0.00000007200000	-0.00000000004159	-1.51525171589879
0.00000007300000	-0.00000000003784	-1.41891601085994
0.00000007400000	-0.00000000003247	-1.25302002629042
0.00000007500000	-0.00000000002563	-1.01807453222036
0.00000007600000	-0.00000000001759	-0.72017484749937
0.00000007700000	-0.00000000000878	-0.37276753436557
0.00000007800000	0.00000000000019	0.00126108536749
0.00000007900000	0.00000000000849	0.36713426279700
0.00000008000000	0.00000000001506	0.67532803833047
0.00000008100000	0.00000000001853	0.85826050744362
0.00000008200000	0.00000000001722	0.82647518628648
0.00000008300000	0.00000000000905	0.46426215681840
0.00000008400000	-0.00000000000848	-0.37534396649941
0.00000008500000	-0.00000000003835	-1.87625512220048

12-30



From conservation of momentum

$$M y_0 = 2m y \qquad \frac{M}{m} = n$$

$$(y + y_0) = \frac{Pl^3}{3EI} = \frac{(m\omega^2 y)\,l^3}{3EI} = y + \frac{2m}{M} y = y\left(1 + \frac{2}{n}\right)$$

$$\therefore\ \omega^2 = \frac{3EI}{Ml^3}(n+2) = \frac{6EI}{Ml^3}\left(1 + \frac{n}{2}\right) \qquad \omega = \sqrt{\frac{6EI}{Ml^3}\left(1 + \frac{n}{2}\right)}$$

12-21 Using Eq. 12.12-5

$$\begin{Bmatrix} -V_3 \\ M_3 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & m\omega^2 \\ \ell & 1 & 0 & m\omega^2\ell \\ 3\alpha\ell & 6\alpha & 1 & m\omega^2 3\alpha\ell \\ \alpha\ell^2 & 3\alpha\ell & \ell & (1+m\omega^2\alpha\ell^2) \end{bmatrix}^2 \begin{Bmatrix} -0 \\ 0 \\ 1 \\ \theta_1 \end{Bmatrix} \qquad \text{where} \quad \alpha = \frac{\ell}{6EI}$$

Calculation can be limited to last 2 rows of last two columns.

$$\left.\begin{aligned} u_{33} + u_{34}\theta_1 &= 0 \\ u_{43} + u_{44}\theta_1 &= 0 \end{aligned}\right\} \text{ or } \quad \underbrace{u_{33}u_{44} - u_{34}u_{43} = 0}_{\text{freq. eq.}}$$

$$\text{For } A^2 = AA = U, \quad u_{ij} = \sum_k a_{ik}\, a_{kj}$$

$$\therefore\; u_{33} = \sum_k a_{3k}\, a_{k3} = 1 + m\omega^2 3\alpha\ell^2$$

$$u_{44} = \sum_k a_{4k}\, a_{k4} = m\omega^2\alpha\ell^2 + m\omega^2 3\alpha\ell^2 + m\omega^2 3\alpha\ell^2 + (1+m\omega^2\alpha\ell^2)^2$$

$$= 1 + 9m\omega^2\alpha\ell^2 + (m\omega^2\alpha\ell^2)^2$$

$$u_{34} = \sum_k a_{3k}\, a_{k4} = m\omega^2 3\alpha\ell + m\omega^2 6\alpha\ell + m\omega^2 3\alpha\ell + m\omega^2 3\alpha\ell(1+m\omega^2\alpha\ell^2)$$

$$= 15\, m\omega^2\alpha\ell + 3(m\omega^2\alpha\ell)^2\ell$$

$$u_{43} = \ell + \ell(1+m\omega^2\alpha\ell^2) = 2\ell + m\omega^2\alpha\ell^3$$

subst. into freq. eq.

$$(1+m\omega^2 3\alpha\ell^2)\left[1+9\omega^2 m\alpha\ell^2 + (m\omega^2\alpha\ell^2)^2\right]$$

$$+ (2\ell + m\omega^2\alpha\ell^3)\left[-15m\omega^2\alpha\ell - 3\ell(m\omega^2\alpha\ell)^2\right] = 0$$

or $\quad 1 - 18(m\omega^2\alpha\ell^2) + 7(m\omega^2\alpha\ell^2)^2 = 0$

let $\beta = m\omega^2\alpha\ell^2$ $\quad$ then $\quad \beta^2 - \frac{18}{7}\beta + \frac{1}{7} = 0$

$$\beta = 1.2857 \pm \sqrt{1.6531 - .1429} = 1.2857 \pm 1.2289$$

$$\beta = \begin{Bmatrix} .0568 \\ 2.5615 \end{Bmatrix} \qquad \sqrt{\beta} = \begin{Bmatrix} .2384 \\ 1.5857 \end{Bmatrix} \qquad \omega = \begin{Bmatrix} .584 \\ 3.884 \end{Bmatrix}\sqrt{\frac{EI}{m\ell^3}}$$

12-31

```
%This pogram uses Myklestad's method on a three mass beam. It is
% set up for evenly spaced lumped masses.
clear; clg;
n=3;
mass=[1 1 1];
l=1;
ei=.1e7;
coef=l/ei;
coef2=(l^2)/(2*ei);
coef3=(l^3)/(3*ei);
for j=1:200
omega(j)=150 +10*j;
va(1)=0;
ma(1)=0;
thetaa(1)=0;
ya(1)=1.;
for i=1:n
 va(i+1)=va(i)-mass(i)*omega(j)^2*ya(i);
 ma(i+1)=ma(i)-va(i+1)*l;
 thetaa(i+1)=thetaa(i)+ma(i+1)*coef+va(i+1)*coef2;
 ya(i+1)=ya(i)+l*thetaa(i)+ma(i+1)*coef2+va(i+1)*coef3;
end
vb(1)=0;
mb(1)=0;
thetab(1)=1;
yb(1)=0;
for i=1:n
 vb(i+1)=vb(i)-mass(i)*omega(j)^2*yb(i);
 mb(i+1)=mb(i)-vb(i+1)*l;
 thetab(i+1)=thetab(i)+mb(i+1)*coef+vb(i+1)*coef2;
 yb(i+1)=yb(i)+l*thetab(i)+mb(i+1)*coef2+vb(i+1)*coef3;
end
theta1=-thetaa(4)/thetab(4);
y4(j)=ya(4)+yb(4)*theta1;
end
[omega' y4']
```

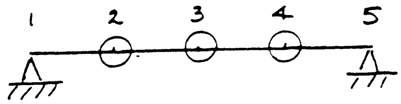
12-31 cont.

omega 1.0e+03 *	y4	
0.1600	0.0006	
0.1700	0.0006	
0.1800	0.0006	
0.1900	0.0005	
0.2000	0.0005	
0.2100	0.0004	
0.2200	0.0004	
0.2300	0.0003	
0.2400	0.0003	
0.2500	0.0002	
0.2600	0.0002	
0.2700	0.0001	
0.2800	0.0001	
0.2900	0.0000	← ω_1
0.3000	-0.0000	
0.3100	-0.0001	
0.3200	-0.0001	
0.3300	-0.0002	
0.3400	-0.0003	
0.3500	-0.0003	
0.3600	-0.0004	
0.3700	-0.0004	
0.3800	-0.0005	
0.3900	-0.0005	
0.4000	-0.0006	
0.4100	-0.0006	
0.4200	-0.0007	
0.4300	-0.0007	
0.4400	-0.0007	

values deleted ↓

omega	y4	
1.7200	-0.0005	
1.7300	-0.0005	
1.7400	-0.0005	
1.7500	-0.0005	
1.7600	-0.0004	
1.7700	-0.0004	
1.7800	-0.0004	
1.7900	-0.0004	
1.8000	-0.0003	
1.8100	-0.0003	
1.8200	-0.0003	
1.8300	-0.0002	
1.8400	-0.0002	
1.8500	-0.0002	
1.8600	-0.0002	
1.8700	-0.0001	
1.8800	-0.0001	
1.8900	-0.0001	
1.9000	-0.0000	
1.9100	-0.0000	← omega$_2$
1.9200	0.0000	
1.9300	0.0000	
1.9400	0.0001	
1.9500	0.0001	
1.9600	0.0001	
1.9700	0.0002	
1.9800	0.0002	

12 - 32



stations must be numbered as 1 to 5 with

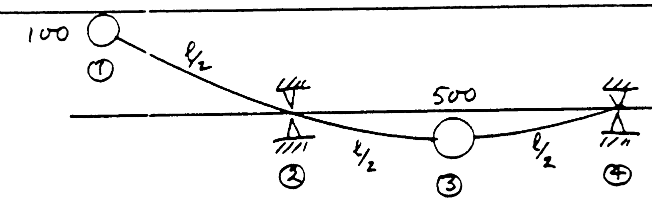
$m_1 = m_5 = 0$

$$\begin{Bmatrix} -V \\ 0 \\ \theta \\ 0 \end{Bmatrix}_5 = \left[\begin{array}{cc|cc} - & - & - & - \\ u_{21} & - & u_{23} & - \\ - & - & - & - \\ u_{41} & - & u_{43} & - \end{array}\right] \begin{Bmatrix} -V \\ 0 \\ \theta \\ 0 \end{Bmatrix}_1$$

$$\therefore \quad \begin{vmatrix} u_{21} & u_{23} \\ u_{41} & u_{43} \end{vmatrix} = 0 \qquad \text{or} \qquad u_{43} = \frac{u_{41}\, u_{23}}{u_{21}}$$

---

12 - 33



$m_2 = m_4 = 0$

$y_1 = 1.0$

$y_2 = y_4 = 0$

Eq. 1, 1-5

$$\begin{Bmatrix} -V \\ M \\ \theta \\ 0 \end{Bmatrix}_2 = \begin{bmatrix} \text{sec } 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \theta_1 \\ 1 \end{Bmatrix}_1 \qquad \begin{Bmatrix} -V \\ M \\ \theta \\ y \end{Bmatrix}_3 = \begin{bmatrix} \text{sec. } 2 \end{bmatrix} \begin{Bmatrix} -V \\ M \\ \theta \\ 0 \end{Bmatrix}_2 \quad \text{etc}$$

this is constraint eq.

$$\begin{Bmatrix} -V \\ 0 \\ \theta \\ 0 \end{Bmatrix}_4 = \begin{bmatrix} 0\ 0 & \\ 0\ \text{sec } 1 & \\ 0\ 0 & \\ 0\ 0 & \end{bmatrix} \begin{bmatrix} & 0 \\ \text{sec } 2 & 0 \\ & 0 \\ & 0 \end{bmatrix} \begin{bmatrix} \text{sec } 3 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \theta_1 \\ 1 \end{Bmatrix}_1 = \begin{bmatrix} - & - & - & - \\ - & - & u_{23} & u_{24} \\ - & - & - & - \\ - & - & u_{43} & u_{44} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \theta_1 \\ 1 \end{Bmatrix}$$

$$\begin{vmatrix} u_{23} & u_{24} \\ u_{43} & u_{44} \end{vmatrix} = 0$$

12-34 Eqs. 12.4-1 to 6 can be arranged in the following matrix form

$$\left\{\begin{array}{c}-V\\ M\\ \theta\\ y\\ \hline T\\ \varphi\end{array}\right\}_{i+1} = \left[\begin{array}{cccc|cc} 1 & 0 & 0 & m\omega^2 & 0 & m\omega^2 c\\ \ell & 1 & 0 & m\omega^2\ell & 0 & m\omega^2 c\ell\\ \frac{\ell^2}{2EI} & \frac{\ell}{EI} & 1 & \frac{m\omega^2\ell^2}{2EI} & 0 & \frac{m\omega^2 c\ell^2}{2EI}\\ \frac{\ell^2}{6EI} & \frac{\ell^2}{2EI} & \ell & \left(1+\frac{m\omega^2\ell^3}{6EI}\right) & 0 & \frac{m\omega^2 c\ell^3}{6EI}\\ \hline 0 & 0 & 0 & m\omega^2 c & 1 & J\omega^2\\ 0 & 0 & 0 & m\omega^2 ch & h & (1+J\omega^2 h)\end{array}\right]_i \left\{\begin{array}{c}-V\\ M\\ \theta\\ y\\ \hline T\\ \varphi\end{array}\right\}_i$$

$$\left\{\begin{array}{c}V\\ \hline W\end{array}\right\}_3 = \left[\begin{array}{c|c} A & 2B\\ \hline 2C & D\end{array}\right]\left[\begin{array}{c|c} A & B\\ \hline C & D\end{array}\right]\left\{\begin{array}{c}V\\ \hline W\end{array}\right\}_1$$

Section 2 differs from Sec 1 by $2c_i$ in place of c_i.

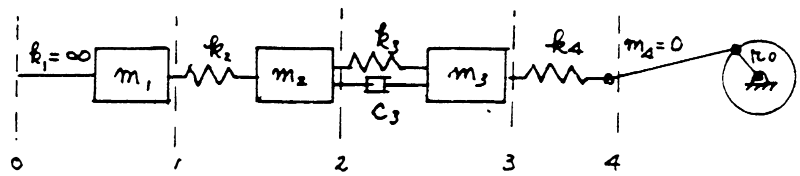
$$= \left[\begin{array}{c|c}(A^2+2BC) & (AB+2BD)\\ \hline (2AC+CD) & (2BC+D^2)\end{array}\right]\left\{\begin{array}{c}V\\ \hline W\end{array}\right\}_1$$

$$\begin{Bmatrix}-V_3\\ M_3\\ 0\\ 0\\ T_3\\ 0\end{Bmatrix} = \begin{bmatrix} & u_{ij} & \end{bmatrix}\begin{Bmatrix}0\\ 0\\ \theta_1\\ 1\\ 0\\ \varphi_1\end{Bmatrix}$$

$$\therefore \begin{bmatrix} u_{33} & u_{34} & u_{36}\\ u_{43} & u_{44} & u_{46}\\ u_{63} & u_{64} & u_{66}\end{bmatrix}\begin{Bmatrix}\theta_1\\ 1\\ \varphi_1\end{Bmatrix} = \begin{Bmatrix}0\\ 0\\ 0\end{Bmatrix}$$

Set determinant to zero for nat. freqs.

12-35



$x_0=1$ $\qquad\qquad$ $x_4 = r_0$

$F_0=0$

$$\begin{Bmatrix} x \\ F \end{Bmatrix}_1 = \begin{Bmatrix} 1 \\ -m_1\omega^2 \end{Bmatrix} = \begin{bmatrix} 1 & \frac{1}{\infty} \\ -m_1\omega^2 & 1-\frac{m_1\omega^2}{\infty} \end{bmatrix}_1 \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}_0 = \begin{Bmatrix} 1 \\ -m_1\omega^2 \end{Bmatrix}$$

$$\begin{Bmatrix} x \\ F \end{Bmatrix}_2 = \begin{bmatrix} 1 & \frac{1}{k_2} \\ -m_2\omega^2 & \left(1-\frac{m_2\omega^2}{k_2}\right) \end{bmatrix}_2 \begin{Bmatrix} x \\ F \end{Bmatrix}_1$$

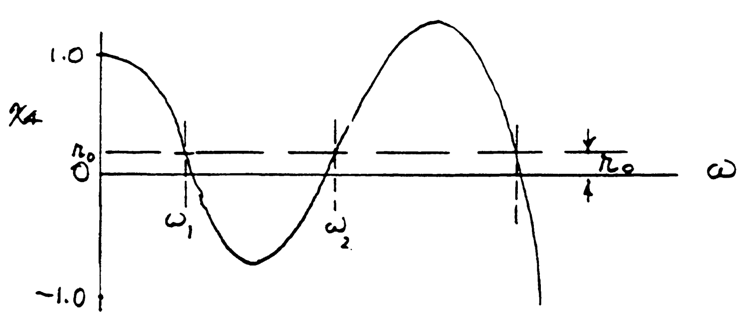
$$\begin{Bmatrix} x \\ F \end{Bmatrix}_3 = \begin{bmatrix} 1 & \frac{1}{k_3+i\omega c_3} \\ -m_3\omega^2 & \left(1-\frac{m_3\omega^2}{k_3+i\omega c_3}\right) \end{bmatrix}_3 \begin{Bmatrix} x \\ F \end{Bmatrix}_2$$

$$\begin{Bmatrix} x \\ F \end{Bmatrix}_4 = \begin{bmatrix} 1 & \frac{1}{k_4} \\ 0 & 0 \end{bmatrix}_4 \begin{Bmatrix} x \\ F \end{Bmatrix}_3$$

Boundry Cond.

$$\therefore\ x_4 = x_3 + \frac{F_3}{k_4} = r_0$$

<u>Problem is identical in calculation to torsional system and same program as Prob 12-3. can be used.</u>



12-36

$$\begin{Bmatrix} \theta \\ T \end{Bmatrix}_0^R = \begin{Bmatrix} 1 \\ -\omega^2 J_1 \end{Bmatrix}$$

$$\begin{Bmatrix} \theta \\ T \end{Bmatrix}_m^R = \begin{bmatrix} 1 & 0 \\ -\omega^2 J & 1 \end{bmatrix}_m \begin{bmatrix} 1 & \frac{1}{K+i\omega g} \\ 0 & 1 \end{bmatrix}_m \begin{Bmatrix} \theta \\ T \end{Bmatrix}_{m-1}^R = \begin{bmatrix} 1 & \frac{1}{K+i\omega g} \\ -\omega^2 J & \left(1 - \frac{\omega^2 J}{K+i\omega g}\right) \end{bmatrix}_m \begin{Bmatrix} \theta \\ T \end{Bmatrix}_{m-1}^R$$

Elements of above transfer matrix are:

$$\frac{1}{K+i\omega g} = \left[\frac{K}{K^2+(\omega g)^2}\right] - i\left[\frac{\omega g}{K^2+(\omega g)^2}\right]$$

$$1 - \frac{\omega^2 J}{K+i\omega g} = \left[1 - \frac{\omega^2 J K}{K^2+(\omega g)^2}\right] + i\left[\frac{\omega^2 J \omega g}{K^2+(\omega g)^2}\right]$$

Sample calc. for $\omega^2 = 0.5 \times 10^5$

Renumber J_s from J_0 to J_3

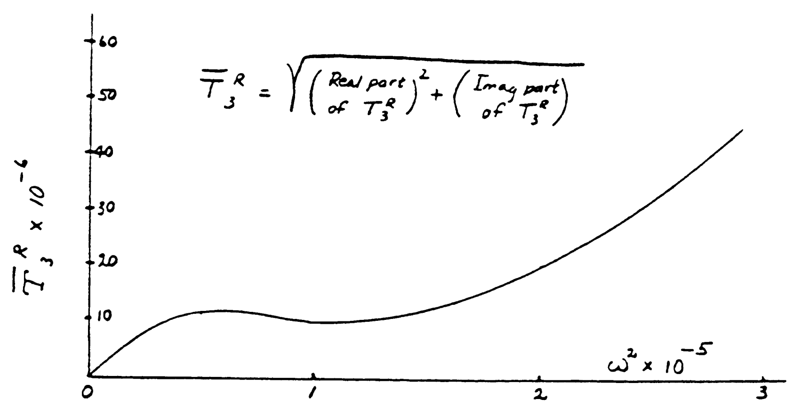
$$\begin{Bmatrix} \theta \\ T \end{Bmatrix}_0^R = \begin{Bmatrix} 1.0 \\ -1 \times 10^6 \end{Bmatrix}$$

$$\begin{Bmatrix} \theta \\ T \end{Bmatrix}_1^R = \begin{bmatrix} 1 & -i(.446 \times 10^{-6}) \\ -5 \times 10^6 & (1 + i\,2.23) \end{bmatrix} \begin{Bmatrix} 1.0 \\ -1 \times 10^6 \end{Bmatrix} = \begin{Bmatrix} 1 + .446\,i \\ (-6 - i\,2.23) \times 10^6 \end{Bmatrix}$$

$$\begin{Bmatrix} \theta \\ T \end{Bmatrix}_2^R = \begin{bmatrix} 1 & .2 \times 10^{-6} \\ -.5 \times 10^6 & .90 \end{bmatrix} \begin{Bmatrix} 1 + .446\,i \\ (-6 - i\,2.23)10^6 \end{Bmatrix} = \begin{Bmatrix} -.20 \\ (-5.9 - i\,2.23)\,10^6 \end{Bmatrix}$$

$$\begin{Bmatrix} \theta \\ T \end{Bmatrix}_3^R = \begin{bmatrix} 1 & .10 \times 10^{-6} \\ -1.5 \times 10^6 & .85 \end{bmatrix} \begin{Bmatrix} -.20 \\ (-5.9 - i\,2.23)10^6 \end{Bmatrix} = \begin{Bmatrix} -.79 - i\,.223 \\ (-4.72 - i\,1.9)\,10^6 \end{Bmatrix}$$

Repeat for other frequencies — should be programmed for digital computer.



12-36. cont.

```
%This program is for problem 12-36 and 12-35
clear,clg;
i=sqrt(-1);
m=4;
L=60;
j=[20 100 10 30 ];
k=1e6*[0 0 5 10];
c=[0 0 0 0 ];
g=[0 1 0 0 ]*1e4;
for ii=1:L
w(ii)=ii*10;
 for jj=1:4
  zr=-w(ii)^2*j(jj);
  zi=w(ii)*c(jj);
  crj(jj)=zr+i*zi;
  zr=k(jj);
  zi=w(ii)*g(jj);
  crk(jj)=zr+i*zi;
end
de(ii,1)=1;
t(ii,1)=crj(1)*de(ii,1);
for jj=2:m
de(ii,jj)=de(ii,jj-1)+t(ii,jj-1)/crk(jj);
t(ii,jj)=crj(jj)*de(ii,jj-1)+(1+crj(jj)/crk(jj))*t(ii,jj-1);
end
de4(ii)=de(ii,4);
t4(ii)=t(ii,4);
end
```

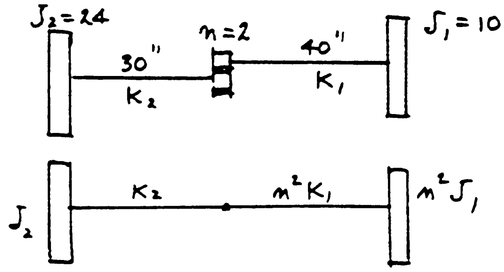
12 - 37

$$\omega^2 = \frac{(J_2 + m^2 J_1) K_2 (m^2 K_1)}{(m^2 J_1)\, J_2\, (K_2 + m^2 K_1)}$$

$$K_1 = G\,\frac{\pi D^4}{32 l} = \frac{(12\times10^6)\pi(1.5)^4}{32\times40} = 0.1492\times10^6$$

$$K_2 = \frac{(12\times10^6)\pi(2)^4}{32\times30} = 0.628\times10^6$$

$$\omega^2 = \frac{(24+40)(.628)(.1492)\times10^{12}}{(240)(.628+.5975)\times10^6} = 2.04\times10^4$$

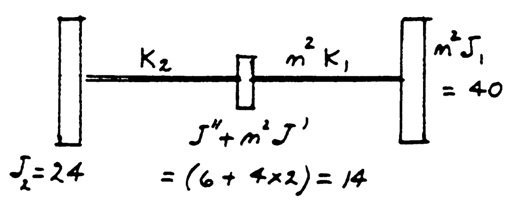


$$\omega = 22.7 \quad Hz$$

---

12 - 38

After reduction to equivalent single shaft system, the freq. eq. of the degenerate 3 - DOF system may be used.



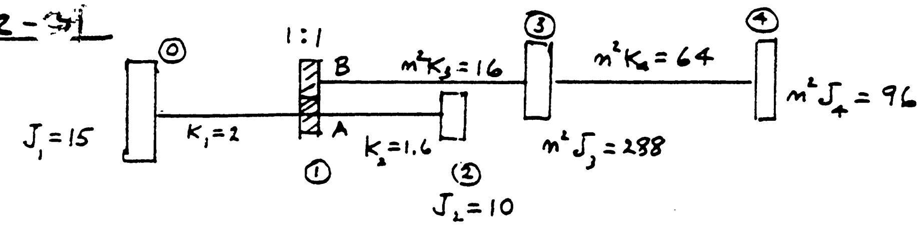
$$\omega^4 - \left[\frac{K_1}{J_1} + \frac{K_2}{J_2}\left(1 + \frac{K_1}{K_2} + \frac{J_2}{J_3}\right)\right]\omega^2 + \frac{K_1}{J_1}\frac{K_2}{J_2}\left(\frac{J_1+J_2+J_3}{J_3}\right) = 0$$

$$\omega^4 - \left[\left(\frac{62.8}{24}\right) + \frac{59.7}{14}\left(1 + \frac{.628}{.597} + \frac{14}{40}\right)\right]10^4\,\omega^2 + \left[\frac{62.8}{24}\cdot\frac{59.7}{14}\left(\frac{24+14+40}{40}\right)\right]10^{3} = 0$$

$$\omega^4 - 12.82\times10^4\,\omega^2 + 21.8\times10^6 = 0$$

$$\omega^2 = \begin{cases} 10.86\times10^4 \\ 2.01\times10^4 \end{cases} \qquad \omega = \begin{cases} 329.2 & r/s \\ 141.8 & " \end{cases} = \begin{cases} 52.30 & Hz \\ 22.55 & " \end{cases}$$

12 - 31



J & K along shaft B are multiplied by $m^2 = 16$ so that the gear ratio is reduced to 1 : 1

Then $\theta_{A1}^{R} = -\theta_{B1}^{R} = \theta_1^{L}$ and $T_{A1}^{R} + T_{B1}^{R} = T_1^{L}$

$$\begin{Bmatrix} \theta \\ T \end{Bmatrix}_0^R = \begin{Bmatrix} 1 \\ -15\omega^2 \end{Bmatrix} = \text{starting eq.}$$

$$\begin{Bmatrix} \theta \\ T \end{Bmatrix}_1^L = \begin{bmatrix} 1 & .5\times10^{-6} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ -15\omega^2 \end{Bmatrix} = \begin{Bmatrix} 1 - 7.5\times10^{-6}\,\omega^2 \\ -15\,\omega^2 \end{Bmatrix}$$

$$\begin{Bmatrix} \theta \\ T \end{Bmatrix}_3^R = \begin{bmatrix} 1 & .0625\times10^{-6} \\ -288\omega^2 & (1-18\times10^{-6}\omega^2) \end{bmatrix} \begin{Bmatrix} \theta \\ T \end{Bmatrix}_{B1}^R$$

$$\begin{Bmatrix} \theta \\ T \end{Bmatrix}_4^R = \begin{bmatrix} 1 & .01562\times10^{-6} \\ -96\omega^2 & (1-1.5\times10^{-6}\omega^2) \end{bmatrix} \begin{Bmatrix} \theta \\ T \end{Bmatrix}_3^R$$

$$\begin{Bmatrix} \theta \\ T \end{Bmatrix}_2^R = \begin{bmatrix} 1 & .625\times10^{-6} \\ -10\omega^2 & (1-6.25\times10^{-6}\omega^2) \end{bmatrix} \begin{Bmatrix} \theta \\ T \end{Bmatrix}_{A1}^R$$

Solution

$$\omega_1 = 377.2 \qquad \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}^{(1)} = \begin{Bmatrix} 1.0 \\ -4.35 \end{Bmatrix}$$

$$\omega_2 = 427.0 \qquad \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}^{(2)} = \begin{Bmatrix} 1.0 \\ 2.605 \end{Bmatrix}$$

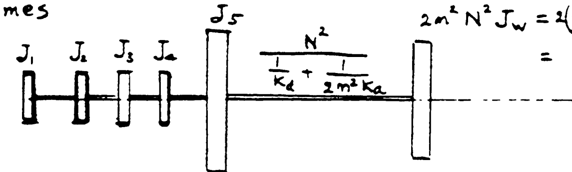
$$\omega_3 = 940.0 \qquad \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}^{(3)} = \begin{Bmatrix} 1.0 \\ 1.725 \end{Bmatrix}$$

12-40

With $n = \frac{1}{3.5}$ the rear wheels and axle are replaced by $2n^2 J_w$ and $2n^2 K_a$. In series with the drive shaft the stiffness is

$$\frac{1}{\frac{1}{K_d} + \frac{1}{2n^2 K_a}}$$

. Let $N = \frac{1}{3}$ = transmission gear ratio, and refer all stiffnesses and inertias to engine speed by multiplying by N^2. System then becomes



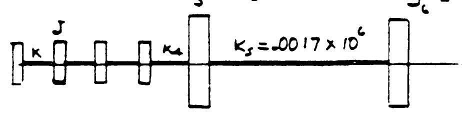
$$2n^2 N^2 J_w = 2\left(\frac{1}{3.5}\right)^2\left(\frac{1}{3}\right)^2 9.2 = .167$$

$$K_a = \frac{(12\times 10^6)\pi(1.25)^4}{32 \times 25} = .115 \times 10^6$$

$$K_d = \frac{(12\times 10^6)\pi(1.5)^4}{32 \times 74} = .0806 \times 10^6$$

$$K_5 = \frac{\left(\frac{1}{3}\right)^2 \times 10^6}{\frac{1}{.0806} + \frac{3.5^2}{.115}} = .00170 \times 10^6$$

12-41



$J = J_1 = J_2 = J_3 = J_4 = 0.20$ lb in sec^2

$K = 6.1 \times 10^6$ lb in/rad $\qquad K_4 = 4.5 \times 10^6$ lb in/rad

Approximation to first nat. freq. as two mass system

$$\omega = \sqrt{\frac{K(J_1 + J_2)}{J_1 J_2}} = 10^3 \sqrt{\frac{.0017(13.1 + .168)}{13.1 \times .168}}$$

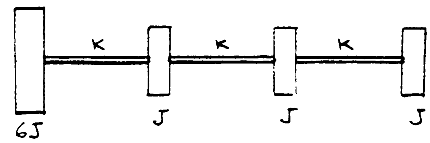
$$= 10^3\sqrt{.0102} \cong 100 \text{ rad/sec}$$

computer solution follows

12-41 cont.

```
%This code is for problem 12-41
clear,clg;
m=2000;
cm=50;
step =20;
j=[.2 .2 .2 .2 12.3 .168];
k=1e6*[6.1 6.1 6.1 4.5 .0017];
for ii=1:m
 tl(ii,1)=0;
 theta(ii,1)=1;
 cmsq=cm^2;
tr(ii,1)=-cmsq*j(1);
for kk=1:5
 L=kk+1
theta(ii,L)=theta(ii,kk)+tr(ii,kk)/k(kk);
tl(ii,L)=tr(ii,kk);
tr(ii,L)=tl(ii,L)-cmsq*j(L)*theta(ii,L);
end
cm=cm+step;
end
```

12-42.



The system is first reduced to the above model

The equations in matrix form are

$$K\begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix} = -\omega^2 J\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix}$$

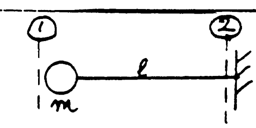
Rearrange to

$$\frac{K}{\omega^2 J}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -.1667 & .1667 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix} = \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix}$$

Results of iteration with sweeping matrix

$\omega_0 = 0, \quad \begin{Bmatrix} 1.0 \\ 1.0 \\ 1.0 \\ 1.0 \end{Bmatrix}$	$\omega_2 = 1.272\sqrt{\frac{K}{J}}, \quad \begin{Bmatrix} 1.00 \\ -.614 \\ -1.237 \\ .1416 \end{Bmatrix}$
$\omega_1 = .5375\sqrt{\frac{K}{J}}, \quad \begin{Bmatrix} 1.00 \\ .714 \\ .239 \\ -.326 \end{Bmatrix}$	$\omega_3 = 1.805\sqrt{\frac{K}{J}}, \quad \begin{Bmatrix} 1.00 \\ -2.27 \\ 1.87 \\ -.1005 \end{Bmatrix}$

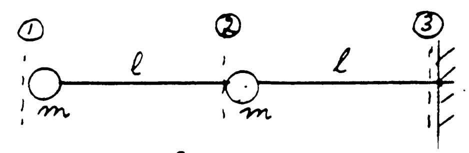
12-43 From Eq. 12.9-5



$$\begin{Bmatrix} -V \\ M \\ 0 \\ 0 \end{Bmatrix}_2 = \left[\begin{array}{cc|cc} - & - & - & - \\ & & 1 & \frac{m\omega^2 \ell^2}{2EI} \\ & & \ell & \left(1+\frac{m\omega^2\ell^3}{6EI}\right) \end{array}\right]\begin{Bmatrix} 0 \\ 0 \\ \theta \\ 1 \end{Bmatrix}_1$$

$$\therefore \begin{vmatrix} 1 & \frac{m\omega^2\ell^2}{2EI} \\ \ell & \left(1+\frac{m\omega^2\ell^3}{6EI}\right) \end{vmatrix} = 0 \qquad \omega = \sqrt{\frac{3EI}{m\ell^3}}$$

12-44



From Eq. 12.9 − 5

Let $\alpha = \ell / 6EI$

$$\begin{Bmatrix} -V \\ M \\ 0 \\ 0 \end{Bmatrix}_3 = \begin{bmatrix} 1 & 0 & 0 & m\omega^2 \\ \ell & 1 & 0 & m\omega^2 \ell \\ 3\alpha\ell & 6\alpha & 1 & 3m\omega^2\alpha\ell \\ \alpha\ell^2 & 3\alpha\ell & \ell & (1+m\omega^2\alpha\ell^2) \end{bmatrix}^2 \begin{Bmatrix} 0 \\ 0 \\ \theta \\ 1 \end{Bmatrix}_1$$

We need only to calculate the last two columns of the last two rows, which are:

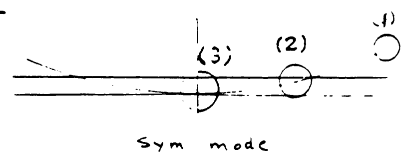
$$\begin{bmatrix} (1+3m\omega^2\alpha\ell^2) & [15m\omega^2\alpha\ell + 3(m\omega^2\alpha)^2\ell^3] \\ (2\ell + m\omega^2\alpha\ell^3) & [1+9m\omega^2\alpha\ell^2 + (m\omega^2\alpha\ell^2)^2] \end{bmatrix} \begin{Bmatrix} \theta \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

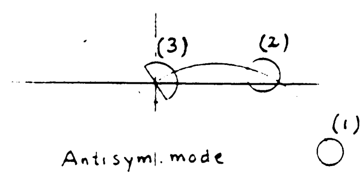
Det. set to zero gives

$$(m\omega^2\alpha\ell^2)^2 - 3(m\omega^2\alpha\ell^2) + \frac{1}{6} = 0$$

solving $\omega = \begin{cases} 0.583 \sqrt{\dfrac{EI}{m\ell^3}} \\ 4.20 \sqrt{\dfrac{EI}{m\ell^3}} \end{cases}$

12-45





$$\begin{Bmatrix} 0 \\ M \\ Q \\ y \end{Bmatrix}_3 = \begin{bmatrix} & & \\ & u_{ij} & \\ & & \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \theta \\ y \end{Bmatrix}_1$$

Freq. eq.

$$\begin{vmatrix} u_{13} & u_{14} \\ u_{33} & u_{34} \end{vmatrix} = 0$$

$$\begin{Bmatrix} -V \\ 0 \\ \theta \\ 0 \end{Bmatrix}_3 = \begin{bmatrix} & & \\ & u_{ij} & \\ & & \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \theta \\ y \end{Bmatrix}_1$$

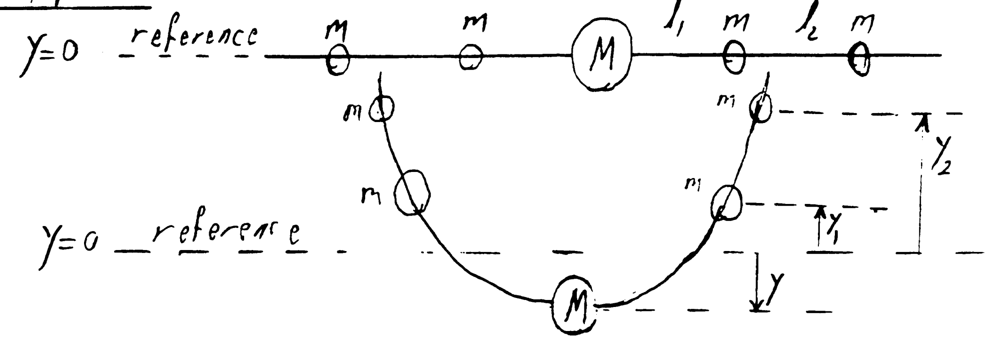
Freq. eq.

$$\begin{vmatrix} u_{23} & u_{24} \\ u_{43} & u_{44} \end{vmatrix} = 0$$

12-46

The solution is the same as problem 12-30.

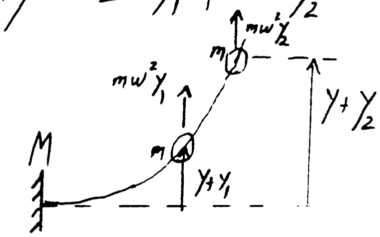
12 - 47



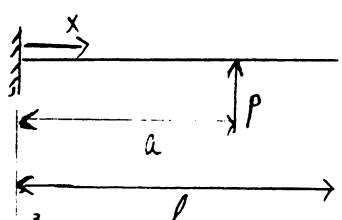
Conservation of momentum gives

$$M\dot{y} = 2m\dot{y}_1 + 2m\dot{y}_2 \xRightarrow[\text{integration}]{} (1)\ My = 2my_1 + 2my_2$$

$$y(x) = \begin{cases} \dfrac{Px^2}{6EI}(3a-x), & 0 \le x \le a \\ \dfrac{Pa^2}{6EI}(3x-a), & a \le x \le l \end{cases}$$



∴ Using the superposition principle



$$(2)\ y+y_1 = \frac{(mw^2y_1)\,l_1^3}{3EI} + \frac{(mw^2y_2)\,l_1^2(3l_2-l_1)}{6EI}$$

$$(3)\ y+y_2 = \frac{(mw^2y_1)\,l_1^2(3l_2-l_1)}{6EI} + \frac{(mw^2y_2)\,l_2^3}{3EI}$$

From (1) and (2) solve for y and y_1 in terms of y_2

$$y = \frac{2}{n}y_1 + \frac{2}{n}y_2$$

(2) becomes $\left(1+\frac{2}{n}\right)y_1 + \frac{2}{n}y_2 = \dfrac{(mw^2y_1)\,l_1^3}{3EI} + \dfrac{(mw^2y_2)\,l_1^2(3l_2-l_1)}{6EI}$

$$\therefore\quad y_1 = \frac{1}{1+\frac{2}{n}-\frac{mw^2l_1^3}{3EI}}\cdot\left[\frac{mw^2l_1^2(3l_2-l_1)}{6EI}-\frac{2}{n}\right]y_2$$

12-47 cont.

$$Y = \left[\frac{2/n}{1 + 2/n - \frac{m w^2 l_1^3}{3EI}} \cdot \left[\frac{m w^2 l_1^2 (3l_2 - l_1)}{6EI} - \frac{2}{n}\right] + \frac{2}{n}\right] Y_2$$

From (3), we get

$$\left[\frac{2/n}{1 + 2/n - \frac{m w^2 l_1^3}{3EI}} \cdot \left[\frac{m w^2 l_1^2 (3l_2 - l_1)}{6EI} - \frac{2}{n}\right] + \frac{2}{n} + 1\right] =$$

$$\frac{m w^2 l_1^2 (3l_2 - l_1)}{6EI} \cdot \frac{1}{1 + \frac{2}{n} - \frac{m w^2 l_1^3}{3EI}} \cdot \left[\frac{m w^2 l_1^2 (3l_2 - l_1)}{6EI} - \frac{2}{n}\right] + \frac{m w^2 l_2^3}{3EI}$$

$$\therefore \frac{2}{n}\left[\frac{m w^2 l_1^2 (3l_2 - l_1)}{6EI} - \frac{2}{n}\right] + \frac{2}{n} + 1 = \frac{m w^2 l_1^2 (3l_2 - l_1)}{6EI}\left[\frac{m w^2 l_1^2 (3l_2 - l_1)}{6EI} - \frac{2}{n}\right] + \left(1 + \frac{2}{n} - \frac{m w^2 l_1^3}{3EI}\right)\frac{m w^2 l_2^3}{3EI}$$

$$\Rightarrow \quad A w^4 + B w^2 + C = 0 \text{ , where ,}$$

$$A = \left[\frac{m l_1^2 (3l_2 - l_1)}{6EI}\right]^2 - \frac{m^2 l_1^3 l_2^3}{(3EI)^2}$$

$$B = \left(1 + \frac{2}{n}\right)\frac{m l_2^3}{3EI} - \frac{2}{n}\,\frac{m l_1^2 (3l_2 - l_1)}{3EI}$$

$$C = \frac{4}{n^2} - \frac{2}{n} - 1$$

$$\therefore w_{1,2}^2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$w_{1,2} = \sqrt{w_{1,2}^2}$ with $w_{1,2}^2 \geq 0$. Ignore $w_{1,2}^2 < 0$

13 - 1

(1) noise in radio reception – wide freq. range & probably stationary.

(2) ground motion of earthquake – non stationary

(3) ocean wave heights – non stationary and dependent on wind or sea state

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13 - 2 See text section 13.2.

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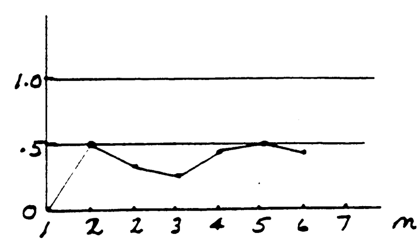
13 - 3 See text sections 13.2-13.3 for definition $= \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} x_i$

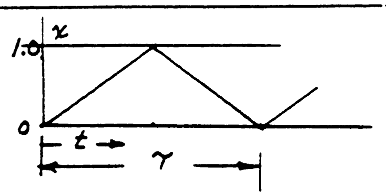
$E(k)$ for	800	coins	thrown	100	times	= 400
" "	8	"	"	"	"	= 4
" "	8	"	"	1000	"	= 4

13-4 Make table as suggested here

Throw	H	T	sum	sum ÷ n
n = 1		0	0	0
2	1		1	1/2
3		0	1	1/3
4		0	1	1/4
5	1		2	2/5
6	1		3	3/6
7		0	3	3/7
etc				



13-5



$x = \frac{2t}{\tau}$ $\quad t \le \tau/2$

$x = 2 - \frac{2t}{\tau}$ $\quad t \ge \tau/2$

Mean value = 0.50

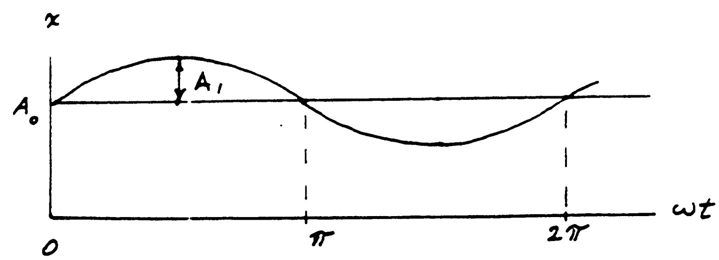
Mean square value

$$\overline{x^2} = \frac{1}{\tau}\left\{ \int_0^{\tau/2} \left(\frac{2t}{\tau}\right)^2 dt + \int_{\tau/2}^{\tau} \left[2 - \frac{2t}{\tau}\right]^2 dt \right\}$$

$$= \frac{1}{\tau}\, \frac{4}{\tau^2}\, \frac{t^3}{3} \Big|_0^{\tau/2} + \left(4t - \frac{8}{\tau}\frac{t^2}{2} + \frac{4}{\tau^2}\frac{t^3}{3}\right)\Big|_{\tau/2}^{\tau}$$

$$= 1/3$$

13 – 6



$x = A_0 + A_1 \sin \omega t$ — Mean val. $= A_0$ by inspection

Let $\omega t = \theta$

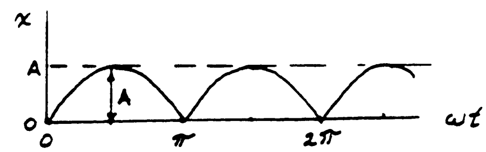
$$\overline{x^2} = \frac{1}{2\pi}\int_0^{2\pi}\left(A_0^2 + 2A_0A_1\sin\theta + A_1^2\sin^2\theta\right) d\theta$$

$$= \frac{1}{2\pi}\left\{A_0^2\theta - 2A_0A_1\cos\theta + \frac{A_1^2}{2}\theta - \frac{A_1^2}{2}\frac{\sin 2\theta}{2}\right\}_0^{2\pi}$$

$$= \frac{1}{2\pi}\left\{A_0^2\, 2\pi - 2A_0A_1(1-1) + \frac{A_1^2}{2}2\pi - 0\right\} = A_0^2 + \frac{1}{2}A_1^2$$

13 – 7

$x = |A \sin \omega t|$



$$\bar{x} = \frac{A}{\pi}\int_0^{\pi}\sin\theta\, d\theta = \frac{A}{\pi}(-\cos\theta)\Big|_0^{\pi} = \frac{2A}{\pi}$$

$$\overline{x^2} = \frac{A^2}{\pi}\int_0^{\pi}\sin^2\theta\, d\theta = \frac{A^2}{\pi}\frac{\theta}{2}\Big|_0^{\pi} = \frac{A^2}{2}$$

13 – 8 Peak values are positive quantities ∴ cannot have a probability in the negative region. Also probability of zero or infinite peaks is zero.

13 – 9 Area under normal probability curve = 1.0

$$\int_{-\infty}^{\infty}\frac{e^{-\frac{x^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}\,dx = \frac{\sigma\sqrt{2\pi}}{\sigma\sqrt{2\pi}} = 1.0 \qquad (a)$$

13-9 Cont.

For higher moments let $\alpha = \frac{1}{2\sigma^2}$ and

$$I = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx \quad \text{- - - - - - -} \quad (b)$$

From (a) $I = \sigma\sqrt{2\pi} = \sqrt{\pi}\,\alpha^{-\frac{1}{2}}$

Start with $I = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\pi}\,\alpha^{-\frac{1}{2}}$ (c)

Differentiate w.r.t. α

$$\frac{\partial I}{\partial \alpha} = -\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = -\frac{1}{2}\sqrt{\pi}\,\alpha^{-3/2} \quad (d)$$

$$= -\frac{1}{2}\sqrt{\pi}\,(2\sigma^2)^{3/2} = -\sqrt{2\pi}\,\sigma^3$$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2 e^{-\alpha x^2}}{\sigma\sqrt{2\pi}} dx = \underline{\underline{\sigma^2 = E(x^2)}}$$

Differentiate w.r.t. α again starting with (d)

$$\int_{-\infty}^{\infty} x^4 e^{-\alpha x^2} dx = -\frac{1}{2}\sqrt{\pi}\left\{-\frac{3}{2}\alpha^{-5/2}\right\} = \frac{3}{4}\sqrt{\pi}\,(2\sigma^2)^{5/2}$$

$$\therefore \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} x^4 e^{-\alpha x^2} dx = \underline{\underline{E(x^4) = 3\sigma^4}}$$

Repeat

$$\frac{\partial}{\partial \alpha}\left\{\frac{3}{4}\sqrt{\pi}\,\alpha^{-5/2}\right\} = -\frac{3}{4}\cdot\frac{5}{2}\sqrt{\pi}\,\alpha^{-7/2} = -\frac{3}{4}\cdot\frac{5}{2}\sqrt{\pi}\,(2\sigma^2)^{7/2}$$

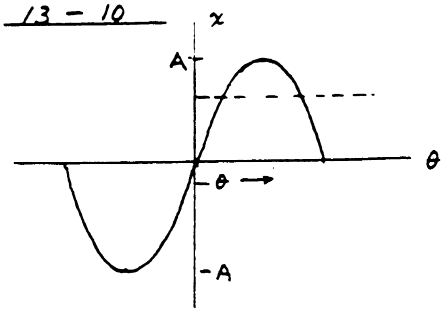
$$= -3\cdot 5\,\sqrt{2\pi}\,\sigma^7$$

$\div$ by $\sigma\sqrt{2\pi}$ $\quad \underline{\underline{E(x^6) = 3\cdot 5\,\sigma^6}}$

General Eq. is $\quad \underline{\underline{E(x^n) = 1\cdot 3\cdot 5\cdot \cdots (n-1)\,\sigma^n}}$ (for n even)

by inspection $E(x^n) = 0$ for n odd.

13 - 10

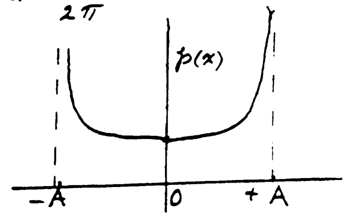


$x = A \sin \theta$

When $x = 0$ $\quad P(x) = \frac{1}{2}$

i.e. half the time x is less than $x = 0$.

As we increase x from $x = 0$ we add $\frac{2\theta}{2\pi}$ to the probability



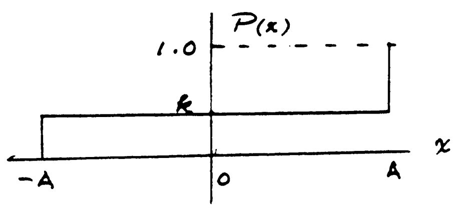
$$\theta = \sin^{-1} \frac{x}{A}$$

$$P(x) = \frac{1}{2} \pm \sin^{-1} \frac{x}{A}$$

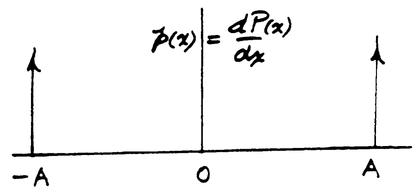
$$p(x) = \frac{d\,P(x)}{dx} = \frac{1}{\pi} \frac{d}{dx} \left(\sin^{-1} \frac{x}{A} \right) = \frac{1}{\pi \sqrt{A^2 - x^2}}$$

---

13 - 11 Measure the total length of line at $-A$ and divide by total length at A and $-A$. Let this fraction be k; then the cumulative prob. curve will appear as in (a). Density curve is shown in (b)



(a)



(b)

---

13 - 12 $\quad x(t) = A \cos t \qquad x(t+\tau) = A \cos(t+\tau)$

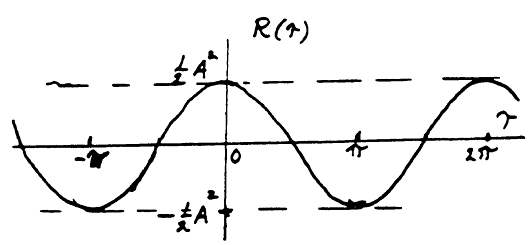
$$= A[\cos t \cos \tau - \sin t \sin \tau]$$

$$x(t)\,x(t+\tau) = A^2 [\cos^2 t \cos \tau - \cos t \sin t \sin \tau]$$

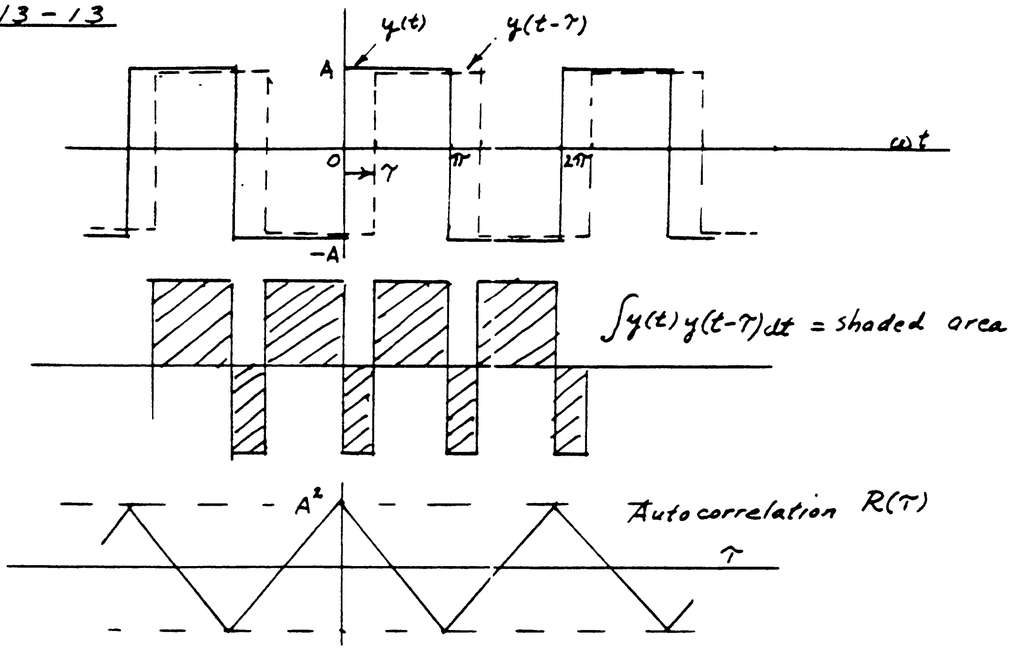
$$R(\tau) = \lim_{T \to \infty} \frac{A^2}{T} \int_{-T/2}^{T/2} [\cos \tau \cdot \tfrac{1}{2}(1 + \cos 2t) - \sin \tau \cdot \sin t \cos t]\, dt$$

$$= \lim_{T \to \infty} \frac{A^2}{T} [\cos \tau \cdot (\tfrac{T}{2}) - \sin \tau \cdot (0)]$$

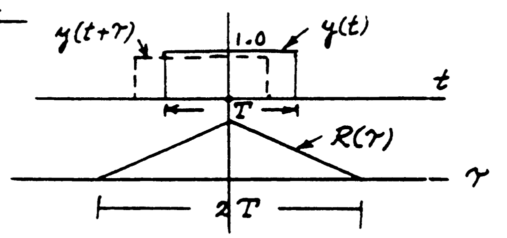
$$= \frac{A^2}{2} \cos \tau$$



13-13

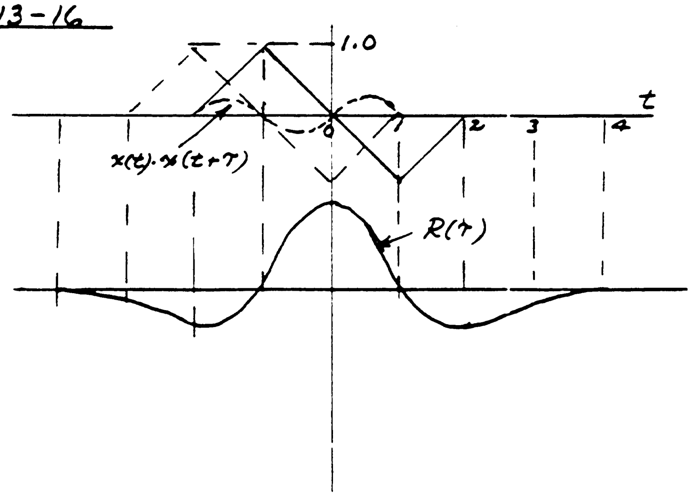


13-14



13-15 This problem is similar to Prob. 13-14. Integrate the area under $y(t) \cdot y(t+\tau)$ curve. Start curve with $R(\tau) = 5$ at $\tau = 0$ and linearly decrease to $R(\tau) = 1.0$ at $\tau = 1$, etc. Shift traced curve as suggested.

13-16



13 - 17

$$\overline{x^2} = .20 \frac{g^2}{Hz} \times 500\ Hz = 100\ g^2$$

$$= 100 \times 9.81^2 = 9623.6$$

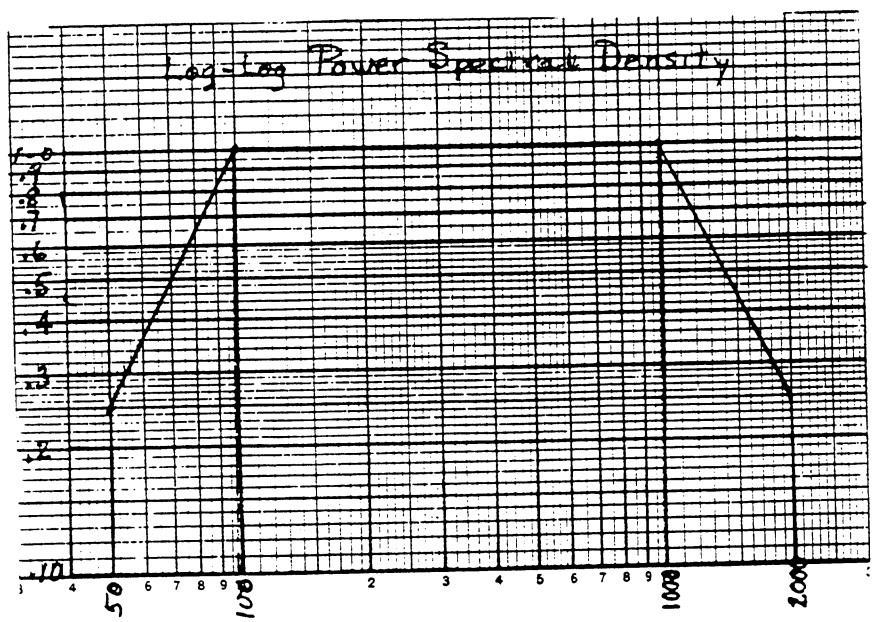
$$RMS = \bar{x} = \sqrt{9623.6} = 98.1 \quad m/s^2$$

13 - 18

$$Area = 1 \times 900 + 2 \times 1000 = 2900\ g^2$$

$$RMS = 9.81 \sqrt{2900} = 528.3 \quad m/s^2$$

13 - 19

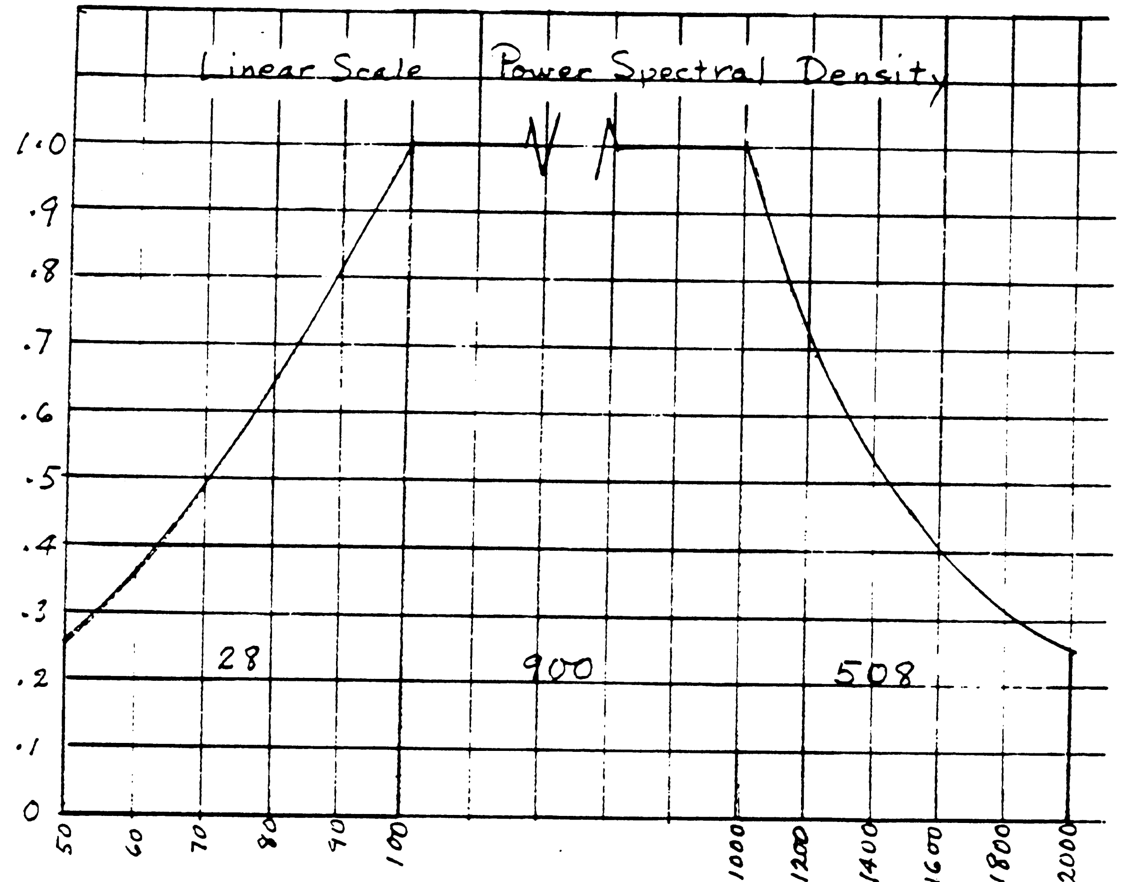


$$DB = 10 \log_{10} \frac{1.0}{.25} = 6.02 \quad db/octave$$

The Log-Log plot is replotted on linear scale

$\overline{x^2}$ = total area = 1436 RMS = 37.9 m/s^2

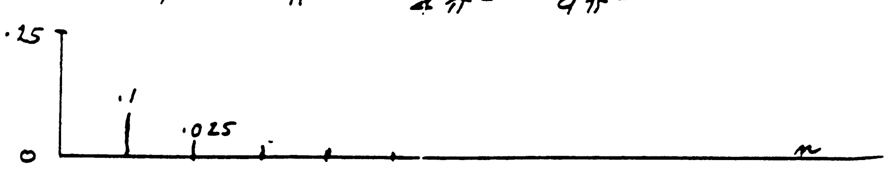
13-19 Cont:



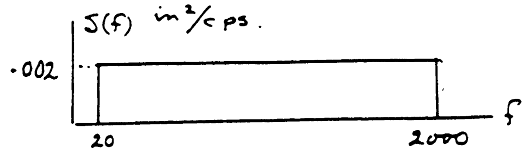
13-20 Procedure here is to determine the Fourier series and plot the quantity $\frac{1}{2} C_m C_m^*$ where $C_n = a_m - i b_m$. Note that $C_m C_m^* = a_m^2 + b_m^2$

i.e. for Fig. 13-20 (a), $x(t) = \frac{1}{2} - \frac{1}{\pi}\left[\sin \omega_1 t + \frac{1}{2} \sin 2\omega_1 t + \cdots\right]$ (see prob. 1-12.) $\therefore a_m = 0$, $b_m = \frac{1}{m\pi}$

$$\sum \frac{1}{2} C_m C_m^* = \frac{1}{4} + \frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} \cdots$$



13 - 21

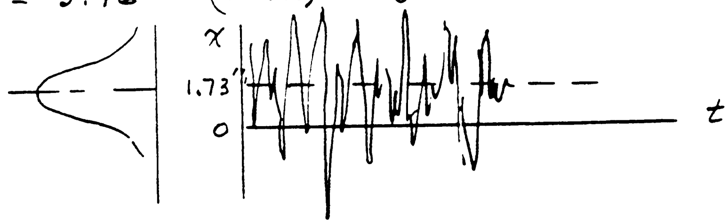


$\overline{x^2} = .002 \times 1980 = 3.96 \ in^2$

$RMS = \sqrt{\overline{x^2}} = 1.99 \ in$ $\qquad \bar{x} = \sqrt{3.0} = 1.732''$

$\sigma^2 = \overline{x^2} - (\bar{x})^2$.

$= 3.96 - (1.732)^2 = 0.9602$ $\qquad \sigma = 0.9799$

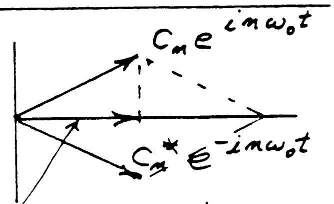


13-22

C_n found by multiplying $f(t)$ by $e^{-in\omega_0 t}$ and integrating over one period

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-in\omega_0 t}\, dt$$

13-23



$C_n = \frac{1}{2}(a_n - i\, b_n)$

$C_0 = \frac{1}{2} a_0$

$$Re\left(C_n e^{in\omega_0 t}\right) = \frac{1}{2}\left(C_n e^{in\omega_0 t} + C_n^* e^{-in\omega_0 t}\right)$$

$$\therefore \ f(t) = \frac{a_0}{2} + \frac{1}{2}\sum_{n=1}^{\infty}\left(C_n e^{in\omega_0 t} + C_n^* e^{-in\omega_0 t}\right)$$

but $C_n^* = C_{-n}$ $\therefore$

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{1}{2} C_n e^{in\omega_0 t} = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}$$

$C_n = 2 c_n$ (see Eq. 13.2-9)

13-24 Same procedure as Prob 13-20

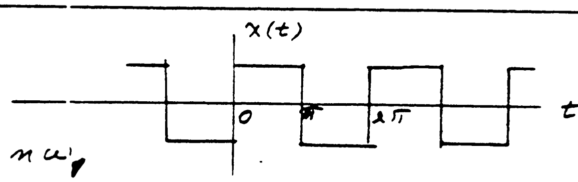
ie Fig P13-24 (See Prob 1-11)

$$x(t) = \frac{1}{2} + \frac{4}{\pi^2}\left(\cos\omega_1 t + \frac{1}{3^2}\cos 3\omega_1 t + \cdots\right)$$

$$S(f) = \sum \frac{C_n C_n^*}{2} = \sum \overline{C_n^2}$$

13-25 See Ch. 1 Sec. 1.2

13-26



$$x = \sum_{n=-\infty}^{\infty} C_n e^{i\omega_n t} \qquad \omega_n = n\omega_1$$

$$C_n = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} x(t)\, e^{-i\omega_n t}\, dt = \frac{1}{2}(a_n - i b_n), \qquad C_0 = \frac{a_0}{2}.$$

Let $\omega_n t = n\omega_1 t = n\theta$ where $\begin{cases} \theta = \omega_1 t \\ d\theta = \omega_1\, dt \end{cases}$

Limits $n\omega_1 \frac{\tau}{2} = n\theta$

$$n\frac{2\pi}{\tau}\frac{\tau}{2} = n\pi = n\theta \qquad \therefore \theta \text{ goes betw. } \pm\pi$$

$$C_n = \frac{1}{\tau}\int_{-\pi}^{\pi} f(\theta)\, e^{-in\theta}\frac{d\theta}{\omega_1} \qquad \omega_1\tau = \frac{2\pi}{\tau}\cdot\tau = 2\pi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, e^{-in\theta}\, d\theta = \frac{2A}{i n\pi} \text{ for } n \text{ odd}$$

$$= 0 \text{ for } n \text{ even}$$

$$\therefore x(t) = \frac{2A}{\pi}\sum_{n=-\infty}^{\infty}\frac{e^{in\omega_1 t}}{in} = \frac{4A}{\pi}\sum_{n=0}^{\infty}\frac{1}{n}\left(\frac{e^{in\omega_1 t} - e^{-in\omega_1 t}}{2i}\right)$$

$$= \frac{4A}{\pi}\left[\sin\omega_1 t + \frac{1}{3}\sin 3\omega_1 t + \frac{1}{5}\sin 5\omega_1 t + \cdots\right]$$

plot $\frac{1}{2}$ (square of ampl.)

13-27 $\left(\frac{Xk}{F}\right) = \frac{1}{\sqrt{(1-\eta^2)^2+(2\zeta\eta)^2}}$ where $\eta = \frac{f}{f_m} = \frac{\omega}{\omega_m}$

At $\eta = 1$ $\frac{Xk}{F} = \frac{1}{2\zeta}$ At half power pt. $\frac{Xk}{F} = \frac{1}{\sqrt{2}}\cdot\frac{1}{2\zeta}$

$$\therefore \frac{1}{2}\left(\frac{1}{2\zeta}\right)^2 = \frac{1}{(1-\eta^2)^2+(2\zeta\eta)^2}$$

$$\eta^4 - 2\eta^2 + 1 + 4\zeta^2\eta^2 = 8\zeta^2$$

$$\eta^4 - 2(1-2\zeta^2)\eta^2 + (1-8\zeta^2) = 0$$

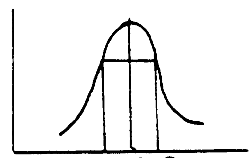
$$\therefore \eta^2 = (1-2\zeta^2) \pm 2\zeta\sqrt{1+\zeta^2}$$

for $\zeta \ll 1$
$\eta^2 \cong 1 \pm 2\zeta$
by neglecting ζ^2

$$\therefore \eta \cong (1 \pm 2\zeta)^{1/2} = 1 \pm \zeta + \text{negl. term}$$

$$\therefore f_1 = f_m(1-\zeta) = f_m\left(1 - \frac{1}{2Q}\right)$$

$$f_2 = f_m(1+\zeta) = f_m\left(1 + \frac{1}{2Q}\right)$$



13-28

$$\int_0^\infty \frac{d\eta}{(1-\eta^2)^2+(2\zeta\eta)^2} \quad \text{where } \eta = \frac{f}{f_m}$$

1st find poles from zeros of denominator

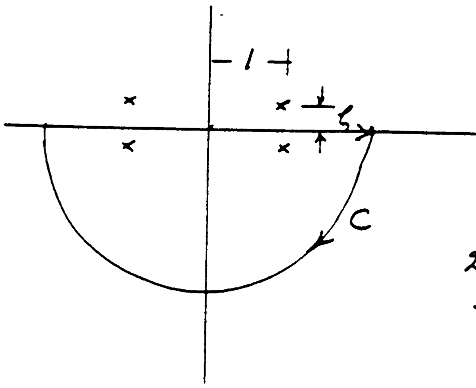
$$F(\eta) = \eta^4 - 2(1-2\zeta^2)\eta^2 + 1 = 0$$

$$\eta^2 = (1-2\zeta^2) \pm i\,2\zeta\sqrt{1-\zeta^2} \cong 1 \pm i2\zeta$$

for $\zeta \ll 1$

See Theory of Residues p 135-137 Laplace Transformation by W. T. Thomson - Prentice-Hall, Inc.

13-28 Cont:



Form contour with infinite circle. Poles are $\eta \cong 1 \pm i\zeta$

Integrate around contour C

$$2\int_0^\infty -- \int_C = 2\pi i \sum \begin{matrix}\text{Residues}\\ \text{within Contour}\end{matrix}$$

Residues at the two poles are

$$\frac{1}{F'(\eta)} = \frac{1}{2(1-\eta^2)(-2\eta)+8\zeta^2\eta} \qquad \text{evaluated at } \eta_1 = 1 - i\zeta$$

$$= -(1+i\zeta)$$

$$\cong i\frac{1}{4\zeta} \qquad \text{and } \eta_1^2 = 1 - i2\zeta$$

$$\eta_2^2 = 1 + i2\zeta$$

Since $\int_C = 0$ we have

$$-2\int_0^\infty = 2\pi i\left(\frac{i}{4\zeta}\right) = -\frac{\pi}{2\zeta}$$

$$\therefore \int_0^\infty \frac{d\eta}{(1-\eta^2)^2+(2\zeta\eta)^2} = \frac{\pi}{4\zeta}$$

Can also be checked by numerical integration (must use very small $\Delta\eta$)

13-29 from definition

$$\bar{X}(s) = \frac{F(s)}{ms^2+k(1+i\gamma)} = \bar{H}(s)\,\bar{F}(s)$$

13-30

$$H(\omega) = \frac{1}{k}\frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\frac{\omega}{\omega_n}\right]^2}}$$

Each component can be treated separately. For 1st component $F\cos(.5\omega_n t-\theta_1)$, the mean square response is

$$\frac{1}{[1-(.5)^2]^2+[.2\times .5]^2}\times\frac{1}{2}\left(\frac{F}{k}\right)^2 = 1.746\times\frac{1}{2}\left(\frac{F}{k}\right)^2$$

similarly other components are

2nd comp $$\frac{1}{[.2]^2}\,\frac{1}{2}\left(\frac{F}{k}\right) = 25\times\frac{1}{2}\left(\frac{F}{k}\right)^2$$

3rd comp $$\frac{1}{[1-2^2]^2+[.2\times 2]^2}\,\frac{1}{2}\left(\frac{F}{k}\right)^2 = .109\times\frac{1}{2}\left(\frac{F}{k}\right)^2$$

$$\therefore\ \overline{x^2} = [1.746+25+.109]\,\frac{1}{2}\left(\frac{F}{k}\right)^2 = 26.85\times\frac{1}{2}\left(\frac{F}{k}\right)^2$$

$$= 13.43\left(\frac{F}{k}\right)^2$$

13-31 From Sec 13.4 & Ex. 13.8-1, $\sigma = 26.6g$

$2\sigma = 53.2g$

$$P[-2\sigma \le x(t) \le 2\sigma] = 95.4\%$$

$$\therefore P[x(t) \ge 2\sigma] = 100-95.4 = \underline{\underline{4.6\%}}$$

$$P[\bar{X} \ge 2\sigma] = \underline{\underline{13.5\%}}$$

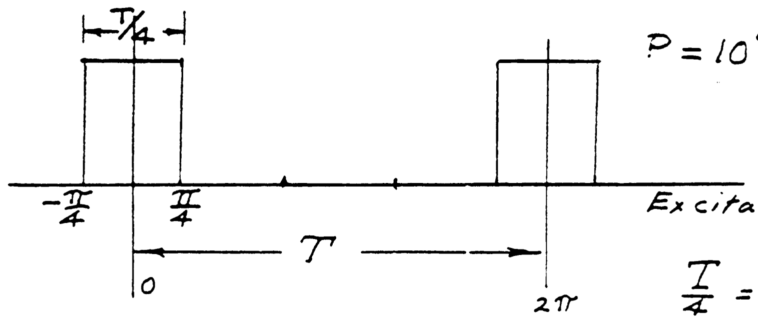
13-32

$m = 40$ Kg. $\quad \omega_n = 2\pi \times 2.20 = 13.82$

Assume $\zeta = .15$ $\quad k = m\omega_n^2 = 40 \times 13.82^2 = 7643.$

$f_n = 2.20$ Hz $\quad \omega_1 = \frac{2\pi}{T}$

$$X = \frac{F_0}{k} \frac{\sin(\omega t - \phi)}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2}}$$



$P = 10^6$ N

Excitation Pulses

$\frac{T}{4} = \frac{2\pi}{4} = \frac{\pi}{2}$

F.S. of the excitation is a cosine series

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\cos n\omega_1 t\, dt = \frac{1}{\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} P\cos n\theta \frac{d\theta}{\omega_1}$$

$$= \frac{P}{\pi\omega_1} \left.\frac{\sin n\theta}{n}\right|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \left(\frac{P}{n\pi\omega_1}\right) 2\sin\frac{n\pi}{4}$$

$$= \left(\frac{2P}{\pi\omega_1}\right)\left(\frac{\sin n\frac{\pi}{4}}{n}\right)$$

$$a_0 = \frac{2}{T}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} P\, dt = \frac{P}{2}$$

n	$\frac{\sin n\frac{\pi}{4}}{n}$
1	$\sqrt{2}/2$
2	$1/2$
3	$\sqrt{2}/6$
4	0
5	$-\sqrt{2}/10$
6	$-1/6$

13-32 Cont.

Excitation Spectrum $S_p(\omega)$

$$S_p(\omega) = \frac{a_o^2}{2} + \sum_{m=1}^{\infty} \frac{a_m^2}{2} = \frac{P^2}{8} + \sum_{m=1}^{\infty} \frac{1}{2}\left(\frac{2P}{\pi\omega_1}\right)^2 \frac{\sin^2 m\frac{\pi}{4}}{m^2}$$

$$= 10^{12}\left[\frac{1}{8} + \left(\frac{2}{\pi^2\omega_1^2}\right)\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{18} + 0 + \frac{1}{50} + \cdots\right)\right]$$

Response Spectrum $\overline{y^2} = \int_0^{\infty} HH^* S_p \, df = \overline{x^2}$

$$= \sum HH^* S_p$$

$$HH^* = \frac{1}{k^2} \frac{1}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2}$$

$\omega = n\omega_1 = n\frac{2\pi}{T}$ $\qquad \omega_n = 2\pi f_n = 2\pi\, 2.2$

$\frac{\omega}{\omega_n} = n\left(\frac{1}{2.2T}\right)$ subst. into $\overline{x^2}$

13-33 For base motion, the relative motion is given by Eq. 3.5-4, which can be written as

$$Z = \frac{\left(\frac{\omega}{\omega_n}\right)^2 Y}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2}}$$

Thus with $S(f)$ = Spectral density of excitation

$$\overline{z^2} = \int_0^{\infty} S_Y(f_+) \frac{\left(\frac{f}{f_n}\right)^4 df}{\left[1-\left(\frac{f}{f_n}\right)^2\right]^2 + \left[2\zeta\frac{f}{f_n}\right]^2} = S_Y(f_n)\cdot f_n \cdot \int_0^{\infty} \frac{\xi^2 d\xi}{\left[1-\xi^2\right]^2 + \left[2\zeta\xi\right]^2}$$

$\therefore$ must evaluate $\int_0^{\infty} \frac{\xi^4 d\xi}{\left[1-\xi^2\right]^2 + \left[2\zeta\xi\right]^2}$

13 - 34

$$\ddot{x} = \frac{(k + i\omega c)}{k - m\omega^2 + i\omega c}\,\ddot{y} = \frac{1 + i\left(25\frac{f}{f_n}\right)}{1 - \left(\frac{f}{f_n}\right)^2 + i\left(25\frac{f}{f_n}\right)}\,\ddot{y}$$

$$\overline{\ddot{x}^2} = \int_0^\infty f_n\,\overline{\ddot{y}^2}\;\frac{1 + i\left(25\frac{f}{f_n}\right)}{1 - \left(\frac{f}{f_n}\right)^2 + i\left(25\frac{f}{f_n}\right)} \cdot \frac{1 - i\left(25\frac{f}{f_n}\right)}{1 - \left(\frac{f}{f_n}\right)^2 - i\left(25\frac{f}{f_n}\right)}\;\frac{df}{f_n}$$

$$= \int_0^\infty S_{\ddot{y}}(f) \cdot f_n \cdot \frac{1 + \left(25\frac{f}{f_n}\right)^2}{\left[1 - \left(\frac{f}{f_n}\right)^2\right]^2 + \left[25\frac{f}{f_n}\right]^2} \cdot d\left(\frac{f}{f_n}\right)$$

---

13 - 35

$$\omega^2 = \frac{k}{m} = \frac{k}{60} = (2\pi \cdot 4)^2$$

$$k = 60 \times (8\pi)^2 = 37{,}899. \;\; N/m$$

$$k^2 = 1436. \times 10^6 \;\; N^2/m^2$$

$$H^2 = \frac{1}{1436 \times 10^6 \left\{\left[1 - \left(\frac{\omega}{8\pi}\right)^2\right]^2 + \left[2(.05)\frac{\omega}{8\pi}\right]^2\right\}}$$

$$\overline{y^2} = \int_0^\infty H^2 S(\omega)\,d\omega \;\cong\; \frac{S(\omega_n)}{k^2}\, f_n\, \frac{\pi}{4\zeta}$$

$$= \frac{100 \times 10^3}{1436 \times 10^6} \cdot \cancel{4} \cdot \frac{\pi}{\cancel{4} \times .05} = .00438$$

$$\sigma^2 = \overline{y^2} = .00438 \;\; m^2, \qquad \sigma = .0662 \;\; m$$

$$y = .132 = 1.99\,\sigma$$

$$P[\,y > 1.99\sigma\,] = \underline{\underline{4.6\,\%}}$$

13 - 36

$m = 272$ kg $f_n = 26$ Hz

$\omega_n = 2\pi f_n = 163.36$

$\zeta = 0.10$

$k = m\omega^2 = 272 \times (163.36)^2$

$= 7258 \times 10^3$ N/m

$k^2 = 52.689 \times 10^{12}$

$\overline{x^2} = \sigma^2 = \frac{S(f_n)}{k^2} \cdot f_n \frac{\pi}{4\zeta}$

$\sigma^2 = \frac{4 \times 10^6}{52.689 \times 10^{12}} \times 26 \times \frac{\pi}{4 \times .10} = 15.50 \times 10^{-6}$

$\sigma = .003937$ $0.012\ m = 3.05\ \sigma$

$P[|x| > 3.05\sigma] = 0.3\%$

13-37

$S(f) = 5 \times 10^6\ N^2/Hz$, $\zeta = 0.03$

$\omega_n = 30$ $f_n = \frac{\omega_n}{2\pi} = 4.775$ Hz

$m = 1500$ kg , $k = m\omega^2 = 1500 \times 30^2 = 1.350 \times 10^6$

$k^2 = 1.823 \times 10^{12}$

$\sigma^2 = \overline{x^2} = \frac{S(f_n)}{k^2} f_n \cdot \frac{\pi}{4\zeta} = \frac{5 \times 10^6}{1.823 \times 10^{12}} \times 4.775 \times \frac{\pi}{4 \times .03}$

$= 342.9 \times 10^{-6}$ $\sigma = 0.01852$

$.037 = 2\sigma$

$P[A > .037] = P[A > 2\sigma] = 13.5\%$

13-38

$$x(t) = \int_0^\infty f(t-\xi)\, g(\xi)\, d\xi$$

$$X(i\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-i\omega t} dt = \int_{-\infty}^{\infty}\int_0^\infty f(t-\xi)\, g(\xi)\, d\xi\, e^{-i\omega t} dt$$

$$= \int_{-\infty}^{\infty}\int_0^\infty f(t-\xi)\, e^{-i\omega(t-\xi)} dt\ \ g(\xi)\, e^{-i\omega\xi} d\xi$$

$$X(i\omega) = \int_0^\infty \left[\int_{-\infty}^{\infty} f(t-\xi)\, e^{-i\omega(t-\xi)} dt\right] g(\xi)\, e^{-i\omega\xi} d\xi$$

Let $(t-\xi) = \tau \qquad dt = d\tau$

$$= \int_0^\infty \left[\int_{-\infty}^{\infty} f(\tau)\, e^{-i\omega\tau} d\tau\right] g(\xi)\, e^{-i\omega\xi} d\xi$$

$$= \int_{-\infty}^{\infty} f(\tau)\, e^{-i\omega\tau} d\tau \int_0^\infty g(\xi)\, e^{-i\omega\xi} d\xi = F(i\omega)\, H(i\omega)$$

$$\overline{x^2} = \int_0^\infty \lim_{T\to\infty} \frac{1}{2\pi T}\, X(i\omega)\, X^*(i\omega)\, d\omega$$

$$= \int_0^\infty \lim_{T\to\infty} \frac{1}{2\pi T}\, F(i\omega)\, F^*(i\omega)\, H(i\omega)\, H^*(i\omega)\, d\omega$$

$$= \int_0^\infty S_F(\omega)\, |H(i\omega)|^2\, d\omega$$

13-39

$$H(i\omega) = |H(i\omega)|\, e^{i\phi(\omega)}$$

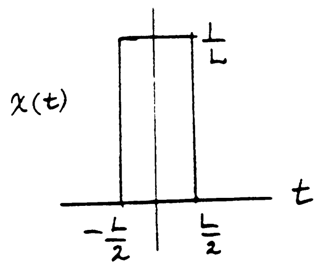
$$H^*(i\omega) = |H(i\omega)|\, e^{-i\phi(\omega)}$$

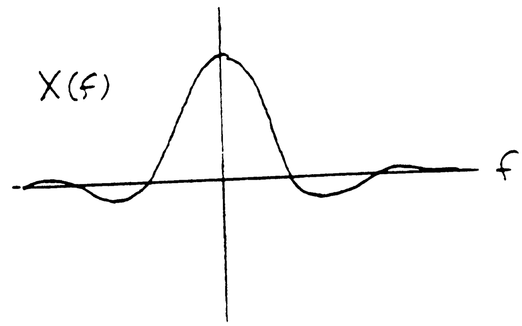
$$\therefore\ \frac{H(i\omega)}{H^*(i\omega)} = e^{i2\phi(\omega)}$$

13-40 Let $X(f)$ = F.T. of rectangular pulse

$$X(f) = \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(\frac{1}{L}\right) e^{-2\pi f t} d = \left(\frac{1}{L}\right) \frac{e^{-i2\pi f t}}{-i2\pi f} \Bigg|_{-\frac{L}{2}}^{\frac{L}{2}}$$

$$= \frac{1}{L} \frac{1}{\pi f} \left[\frac{e^{i2\pi f \frac{L}{2}} - e^{-i2\pi f \frac{L}{2}}}{2i} \right] = \frac{\sin \pi f L}{\pi f L}$$





13-41

$$\int_{-\infty}^{\infty} | f(t) | \, dt = \int_{-\infty}^{\infty} u(t) \, dt = \int_{0}^{\infty} 1 \, dt = \infty$$

$\therefore$ Unit step function cannot have a F.T.

13-42

$$S_{FX}(\omega) = \lim_{T \to \infty} \frac{1}{2\pi T} F^*(i\omega) X(i\omega)$$

$$= \left[\lim_{T \to \infty} \frac{1}{2\pi T} F^*(i\omega) F(i\omega) \right] H(i\omega) = S_F(i\omega) H(i\omega)$$

$$S_{XF}(\omega) = \lim_{T \to \infty} \frac{1}{2\pi T} X^* F = \lim_{T \to \infty} \frac{1}{2\pi T} F^* H^* F = S_F(i\omega) H^*(i\omega)$$

$$\frac{S_{FX}}{S_{XF}} = \frac{S_F |H| e^{i\phi}}{S_F |H| e^{-i\phi}} = e^{i2\phi(\omega)}$$

13-42 Cont:

$$\frac{S_F}{S_{xF}} = \frac{\cancel{S_x}}{\cancel{S_x} H^*} \qquad \frac{S_{Fx}}{S_F} = \frac{\cancel{S_F} H}{\cancel{S_F}} = H$$

$$\frac{S_F}{S_{xF}} = \frac{1}{a-ib} = \frac{(a+ib)}{(a^2+b^2)} = \frac{H(i\omega)}{|H|^2}$$

13-43

$$\frac{d^2\bar{u}(x,s)}{dx^2} = \left(\frac{s}{c}\right)^2 \bar{u}(x,s)$$

$$\bar{u}(x,s) = C_1 e^{\frac{sx}{c}} + C_2 e^{-\frac{sx}{c}}$$

$$\bar{F}(x,s) = AE\frac{d\bar{u}}{dx} = AE\frac{s}{c}\left[C_1 e^{\frac{sx}{c}} - C_2 e^{-\frac{sx}{c}}\right]$$

$$\bar{F}(0,s) = AE\frac{s}{c}\left[C_1 - C_2\right]$$

$$\bar{F}(\ell,s) = 0 = C_1 e^{\frac{s\ell}{c}} - C_2 e^{-\frac{s\ell}{c}} \qquad \therefore \quad C_1 = C_2 e^{-\frac{2s\ell}{c}}$$

$$C_2 = \frac{-c\bar{F}(0,s)}{AEs\left(1-e^{-\frac{2s\ell}{c}}\right)}$$

$$\bar{u}(x,s) = \frac{-c\bar{F}(0,s)}{AEs\left(1-e^{-\frac{2s\ell}{c}}\right)}\left[e^{-\frac{2s\ell}{c}} e^{\frac{sx}{c}} + e^{-\frac{sx}{c}}\right]$$

$$= \frac{-c\bar{F}(0,s)\,e^{-\frac{s\ell}{c}}}{AEs\left(1-e^{-\frac{2s\ell}{c}}\right)}\left[e^{\frac{s}{c}(x-\ell)} + e^{-\frac{s}{c}(x-\ell)}\right]$$

13-44

$$p(x,t) = F_0 e^{i\omega t}\delta(x)$$

$$\bar{p}(x,s) = \frac{F_0}{s-i\omega}\delta(x), \qquad \bar{F}(0,s) = \int_0^\ell \bar{p}(x,s)\,dx = \frac{F_0}{s-i\omega}$$

$$\bar{u}(x,s) = \frac{-cF_0\,e^{\frac{s\ell}{c}}}{s(s-i\omega)AE\left(1-e^{-\frac{2s\ell}{c}}\right)}\left[e^{\frac{s}{c}(x-\ell)} + e^{-\frac{s}{c}(x-\ell)}\right]$$

$$\therefore \; u(x,t) = \frac{-cF_0\,e^{i\omega t}}{\omega AE\sin\frac{\omega\ell}{c}}\cos\frac{\omega\ell}{c}\left(\frac{x}{\ell}-1\right)$$

$$\sigma = E\frac{du}{dx} = \frac{-cF_0e^{i\omega t}}{\omega A\sin\frac{\omega\ell}{c}}\cdot\frac{\omega\ell}{c\ell}\sin\frac{\omega\ell}{c}\left(\frac{x}{\ell}-1\right) = \frac{-F_0e^{i\omega t}}{A\sin\frac{\omega\ell}{c}}\sin\frac{\omega\ell}{c}\left(\frac{x}{\ell}-1\right)$$

13-45

$$u(x,t) = \sum_{n=0}^{\infty} \phi_n(x) q_n(t)$$

$$\sigma(x,t) = E\frac{du}{dx} = E\sum_{n=1}^{\infty} \phi_n'(x) q_n(t) \qquad \text{where } \phi' = \frac{d\phi}{dx}$$

$$\overline{\sigma(x,t)\sigma(x',t)} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \sigma(x,t)\sigma(x',t)\,dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{1}{2\pi T} \sigma^*(x,i\omega)\sigma(x',i\omega)\,d\omega$$

$$\sigma(x,i\omega) = E\sum Q_n(i\omega)\phi_n'(x)$$

$$\overline{\sigma(x,t)\sigma(x',t)} = \frac{E^2}{2}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \phi_n'(x)\phi_k'(x')\int_{-\infty}^{\infty} \lim_{T\to\infty}\frac{1}{2\pi T} Q_n^*(i\omega) Q_k(i\omega)\,d\omega$$

$$\ddot{q}_n + \omega_n^2(1+i\gamma)q_n = \frac{1}{m\ell}\int_0^{\ell} p(x,t)\delta(x)\phi_n(x)\,dx = \frac{1}{m\ell}F(0,t)\phi_n(0)$$

$$Q_n(i\omega) = \frac{F(0,i\omega)\phi_n(0)}{m\ell\left[(\omega_n^2-\omega^2)+i\gamma\omega_n^2\right]} = \frac{F(0,i\omega)\phi_n(0)}{m\ell\omega_n^2\left[1-\left(\frac{\omega}{\omega_n}\right)^2+i\gamma\right]}$$

$$\int_{-\infty}^{\infty} \lim_{T\to\infty}\frac{1}{2\pi T}\,\frac{F^*(0,i\omega)F(0,i\omega)\phi_n(0)\phi_k(0)\,d\omega}{m^2\ell^2\omega_n^2\omega_k^2\left[1-\left(\frac{\omega}{\omega_n}\right)^2-i\gamma\right]\left[1-\left(\frac{\omega}{\omega_n}\right)^2+i\gamma\right]}$$

$$= \int_{-\infty}^{\infty} \frac{S(i\omega)\phi_n(0)\phi_k(0)\,d\omega}{m^2\ell^2\omega_n^2\omega_k^2\left[1-\left(\frac{\omega}{\omega_n}\right)^2-i\gamma\right]\left[1-\left(\frac{\omega}{\omega_n}\right)^2+i\gamma\right]}$$

Greatest contribution occurs when $k=n$ $\therefore$ change double Σ to single summation with $k=n$.

13-45 Cont:

$$\overline{\sigma(x,t)\sigma(x',t)} = \frac{E^2}{2}\sum_n \phi_n'(x)\phi_n'(x')\phi_n^2(0)\frac{1}{m^2\ell^2\omega_n^4}\int_{-\infty}^{\infty}\frac{S(\omega)\,d\omega}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\gamma^2}$$

$$\overline{\sigma^2(x,t)} = \frac{E^2}{2}\sum_n \phi_n'^2(x)\phi_n^2(0)\frac{1}{m^2\ell^2\omega_n^3}S(\omega_n)\frac{\pi}{\gamma}$$

where:

$$\phi_n = \sqrt{2}\cos n\pi\left(\frac{x}{\ell}-1\right)$$

$$\omega_n = n\pi\frac{c}{\ell}$$

$$c = \sqrt{\frac{AE}{m}}$$

$$E = \frac{c^2 m}{A}$$

$$\therefore \overline{\sigma^2(x,t)} \cong \frac{2\pi}{\gamma}\sum\frac{c}{A^2 n\pi\ell}S(\omega_n)\sin^2\frac{n\pi x}{\ell}$$

13 - 46

Prove $FT[x(t-t_0)] = e^{-i2\pi f t_0} X(f)$

where $X(f) = FT[x(t)]$

From Eq.(13.6-1) $x(t-t_0) = \int_{-\infty}^{\infty} X(f)\, e^{i2\pi f(t-t_0)} df$

$$= \int_{-\infty}^{\infty} \left[e^{-i2\pi f t_0} X(f) \right] e^{i2\pi f t}\, df$$

Comparison with Eq.(13.6-2) shows $e^{-i2\pi f t_0} X(f) = FT[x(t-t_0)]$

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13 - 47

Prove $FT[x(t) * y(t)] = X(f)Y(f)$

$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau$ = convolution of $x(t) * y(t)$

From Eq(13.6-2)

$$FT[x(t) * y(t)] = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau \right] e^{-i2\pi f t}\, dt$$

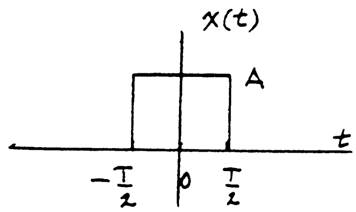
$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} y(t-\tau)\, e^{-i2\pi f t}\, dt \right] x(\tau)\, d\tau$$

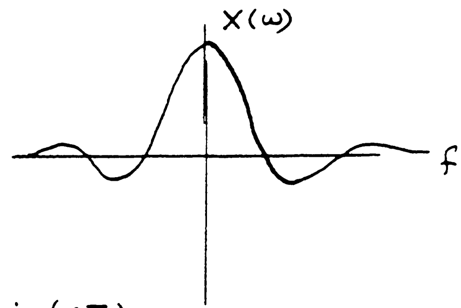
Let $(t-\tau) = \xi$, $t = \xi + \tau$ $dt = d\xi$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} y(\xi)\, e^{-i2\pi f \xi}\, d\xi \right] x(\tau)\, e^{-i2\pi f \tau}\, d\tau$$

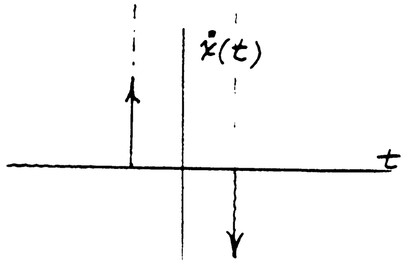
$$= \int_{-\infty}^{\infty} x(\tau)\, e^{-i2\pi f \tau}\, d\tau \cdot \int_{-\infty}^{\infty} y(\xi)\, e^{-i2\pi f \xi}\, d\xi = X(f)\, Y(f)$$

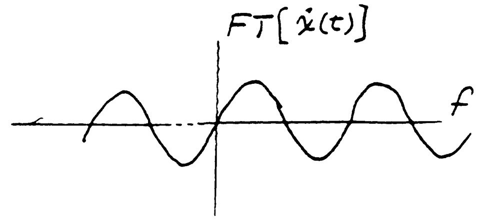
13-48





$$X(\omega) = FT[x(t)] = A\,T\,\frac{\sin\left(\frac{\omega T}{2}\right)}{\left(\frac{\omega T}{2}\right)}$$





$$FT[\dot{x}(t)] = i\omega\,FT[x(t)] = i\cancel{\omega}\cdot A\cancel{T}\,\frac{\sin\left(\frac{\omega T}{2}\right)}{\frac{\cancel{\omega}\cancel{T}}{2}}$$

$$= i\,2A\,\sin\left(\frac{\omega T}{2}\right) = \text{sine wave}$$

13-49

$$\int_{-\infty}^{\infty} \frac{a}{b+x^2}\,dx = 1 \quad (\text{Probability density function})$$

i) if $b>0$, $\displaystyle\int_{-\infty}^{\infty} \frac{a}{b+x^2}\,dx = \frac{a}{\sqrt{b}} \arctan \frac{x}{\sqrt{b}} \Big|_{-\infty}^{\infty}$

$$= \frac{a}{\sqrt{b}}\pi = 1 \Rightarrow a = \frac{\sqrt{b}}{\pi} \quad (b>0)$$

ii) if $b<0$, $\displaystyle\frac{a}{x^2+b} = \frac{a}{2\sqrt{-b}}\left[\frac{-1}{x+\sqrt{-b}} + \frac{1}{x-\sqrt{-b}}\right]$

$$\therefore \int_{-\infty}^{\infty} P(x)dx = \frac{a}{2\sqrt{-b}}\left[\ln\left(\frac{x-\sqrt{-b}}{x+\sqrt{-b}}\right)\Big|_{-\infty}^{\infty}\right]$$

$$= \frac{a}{2\sqrt{-b}}\left[\ln\left(\frac{x-\sqrt{-b}}{x+\sqrt{-b}}\right)\Big|_{-\infty}^{-\sqrt{-b}} + \ln\left(\frac{x-\sqrt{-b}}{x+\sqrt{-b}}\right)\Big|_{\sqrt{-b}}^{\infty}\right]$$

$$= \frac{a}{2\sqrt{-b}}\left[\infty - 0 + 0 - (-\infty)\right] \longrightarrow \pm\infty \text{ if } a \neq 0$$

if $a=0$, $\int_{-\infty}^{\infty} P(x)\,dx = 0$

$\therefore$ Not a probability density function if $b<0$

iii) if $b=0$, $\int_{-\infty}^{\infty} P(x)dx = \begin{cases} 0 & \text{if } a=0 \\ \pm\infty & \text{if } a\neq 0 \end{cases}$

$\therefore$ Not a probability density function

$\therefore$ Answer $a = \dfrac{\sqrt{b}}{\pi}$ $(b>0)$.

13-50

$$P(x) = \begin{cases} 1 - \exp(-\lambda x) & x > 0 \\ 0 & x < 0 \end{cases} \quad , \quad \lambda > 0$$

pdf: $$p(x) = \frac{dP}{dx} = \begin{cases} \lambda \exp(-\lambda x) & x > 0 \\ 0 & x < 0 \end{cases}$$

mean ; $$\bar{x} = \int_0^\infty \lambda x \exp(-\lambda x)\, dx$$

$$= \frac{-\lambda x - 1}{\lambda} e^{-\lambda x} \Big|_0^\infty = \frac{1}{\lambda}$$

variance : $$\sigma^2 = \int_0^\infty \lambda x^2 \exp(-\lambda x)\, dx$$

$$= -x^2 e^{-\lambda x} + \frac{2(-\lambda x - 1)}{\lambda^2} e^{-\lambda x} \Big|_0^\infty$$

$$= \frac{2}{\lambda^2}$$

13 - 51

From problem 5-65

$$\ddot{x} - l\ddot{\theta}\sin\theta - l\dot{\theta}^2\cos\theta + \frac{K}{m}x = 0$$

$$l\ddot{\theta} - \ddot{x}\sin\theta + g\sin\theta = 0$$

Neglecting the motion in the vertical x direction and linearizing the motion in the θ direction, we get

$$l\ddot{\theta} + g\theta = 0$$

Let the wind force be equal to mu. Then

$$l\ddot{\theta} + g\theta = u$$

$$\xrightarrow{\text{Laplace transform}} \quad (ls^2 + g)\,\Theta(s) = U(s) \Longrightarrow \Theta(s) = \frac{1}{ls^2+g}U(s)$$

u has white-noise characteristics.

Assume its mean and variance are W and σ^2, respectively

The mean of the output θ is

$$w_\theta = \left|\frac{1}{l(j0)^2+g}\right| \cdot W = \frac{W}{g}$$

The variance of the output θ is

$$\sigma_\theta^2 = \left|\frac{1}{l(j0)^2+g}\right|^2 \cdot \sigma^2 = \frac{\sigma^2}{g^2}$$

Assuming Gaussian distribution, the pdf for θ is

$$p(\theta) = \frac{1}{\sqrt{2\pi}\,\sigma_\theta}\exp\left[-\frac{1}{2}\left(\frac{\theta - w_\theta}{\sigma_\theta}\right)^2\right]$$

$$\therefore \quad P\left[\theta \leqslant 10^\circ = \frac{\pi}{18}\right] = \int_{-\infty}^{\pi/18} \frac{1}{\sqrt{2\pi}\,\sigma_\theta}\exp\left[-\frac{1}{2}\left(\frac{\theta-w_\theta}{\sigma_\theta}\right)^2\right]$$

$$= \frac{1}{\sigma_\theta}\left[erf\left(\frac{\pi/18 - w_\theta}{\sigma_\theta}\right) - erf\left(\frac{-\infty - w_\theta}{\sigma_\theta}\right)\right],$$

where, $erf(x) = \frac{1}{\sqrt{2\pi}}\int_0^x \exp(-\frac{1}{2}x^2)\,dx$

14-1 If $x_1 = \varphi_1(t)$ and $x_2 = \varphi_2(t)$ are solutions of the equation $\ddot{x} + x^3 = 0$, then they will satisfy

$$\ddot{\varphi}_1 + \varphi_1^3 = 0 \quad \text{and} \quad \ddot{\varphi}_2 + \varphi_2^3 = 0$$

Adding, the following is also true

$$(\ddot{\varphi}_1 + \ddot{\varphi}_2) + (\varphi_1^3 + \varphi_2^3) = 0 \qquad (a)$$

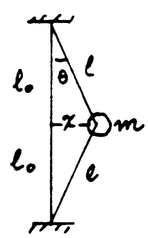
If we assume $x = \varphi_1 + \varphi_2$ and substitute into the D.E. we would obtain

$$(\ddot{\varphi}_1 + \ddot{\varphi}_2) + (\varphi_1^3 + 3\varphi_1^2\varphi_2 + 3\varphi_1\varphi_2^2 + \varphi_2^3) = 0$$

which does not agree with the correct result (a)
Superposition of solutions are in general not solutions of nonlinear equations.

14-2

$$\sum F_x = -2T\sin\theta = m\ddot{x}$$



$$\sin\theta = \frac{x}{\ell} = \frac{x}{\sqrt{\ell_0^2 + x^2}} \cong \frac{x}{\ell_0}\left[1 - \frac{1}{2}\left(\frac{x}{\ell_0}\right)^2\right]$$

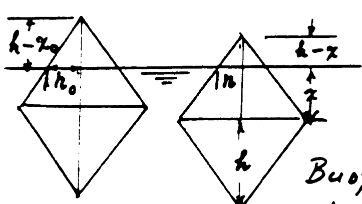
$$T = T_0 + k(\ell - \ell_0) = T_0 + k\left[\ell_0\left(1 + \frac{x^2}{\ell_0^2}\right)^{1/2} - \ell_0\right]$$

$$\cong T_0 + k\,\frac{1}{2}\left(\frac{x}{\ell_0}\right)^2$$

$$\therefore \quad m\ddot{x} + 2\left[T_0 + \frac{k}{2}\left(\frac{x}{\ell_0}\right)^2\right]\frac{x}{\sqrt{\ell_0^2 + x^2}} = 0$$

$$m\ddot{x} + \frac{2}{\ell_0}\left[T_0 + \frac{k}{2}\left(\frac{x}{\ell_0}\right)^2\right]\left[1 - \frac{1}{2}\left(\frac{x}{\ell_0}\right)^2\right]x = 0$$

14-3



Vol. of cone $= \frac{1}{3}\pi r^2(h-x)$

From similar triangles $\frac{r_0}{h-x_0} = \frac{r}{h-x}$

$\therefore r = r_0\left(\frac{h-x}{h-x_0}\right)$

Difference in vol. $= \frac{1}{3}\pi\left[r^2(h-x) - r_0^2(h-x_0)\right]$

Buoyant force $= \rho\,\Delta V = \frac{\pi}{3}\rho r_0^2\left[\left(\frac{h-x}{h-x_0}\right)^2(h-x) - (h-x_0)\right]$

= weight of water displaced.

$$\therefore \; m\ddot{x} = \frac{\pi}{3}\rho r_0^2\left[\frac{(h-x)^3}{(h-x_0)^2} - (h-x_0)\right]$$

$$= \frac{\pi}{3}\rho\frac{r_0^2}{(h-x_0)^2}\left\{(h-x)^3 - (h-x_0)^3\right\}$$

14-4 For $x > x_0$ Eq. of motion is

$$m\ddot{x} + k(x - x_0) = 0 \qquad \text{- - - - - - (a)}$$

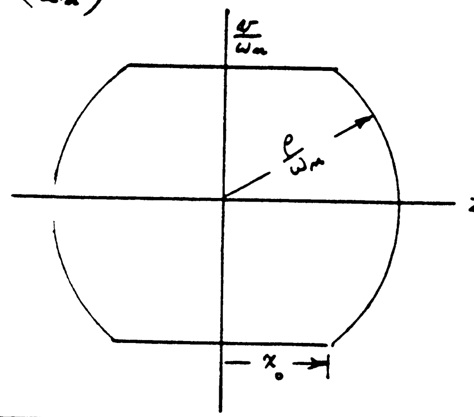
For $x < x_0$ $\quad \ddot{x} = \dot{x}\frac{d\dot{x}}{dx} = 0 \quad \therefore \frac{dv}{dx} = 0 \quad$ where $v = \dot{x}$

Let $z = (x - x_0)$ in Eq. (a); then

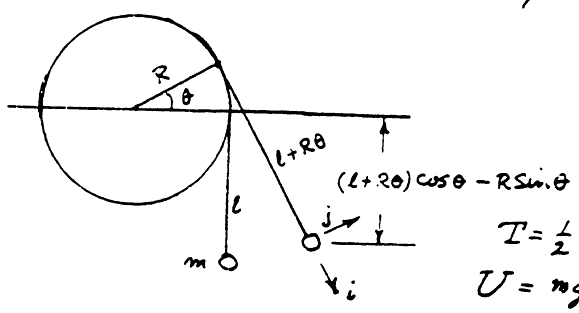
$$\ddot{z} + \omega_n^2 z = 0 \qquad \text{or} \qquad \dot{z}\frac{d\dot{z}}{dz} + \omega_n^2 z = 0$$

Integrating $\quad \dot{z}^2 + \omega_n^2 z^2 = \left(\frac{c}{\omega_n}\right)^2 \quad$ a circle

Phase-plane trajectory will look like this



14-5



Since system is conservative

$$\frac{d}{dt}(T + U) = 0 \qquad \text{(a)}$$

Vel. of m is

$$\vec{v} = (l + R\theta)\dot{\theta}\,\vec{j}$$

$$T = \frac{1}{2}m\vec{v}\cdot\vec{v} = \frac{1}{2}m\dot{\theta}^2(l + R\theta)^2$$

$$U = mg\left[l - (l + R\theta)\cos\theta + R\sin\theta\right]$$

Subst. into (a)

$$\dot{\theta}\left\{\ddot{\theta}(l+R\theta)^2 + \dot{\theta}(l+R\theta)R + g(l+R\theta)\sin\theta - \cancel{R\cos\theta} + \cancel{R\cos\theta}\right\} = 0$$

$$\ddot{\theta} + \frac{R\dot{\theta}^2}{l+R\theta} + \frac{g}{l+R\theta}\sin\theta = 0$$

14-6 $\quad \ddot{x} + \omega_m^2 x = 0$,

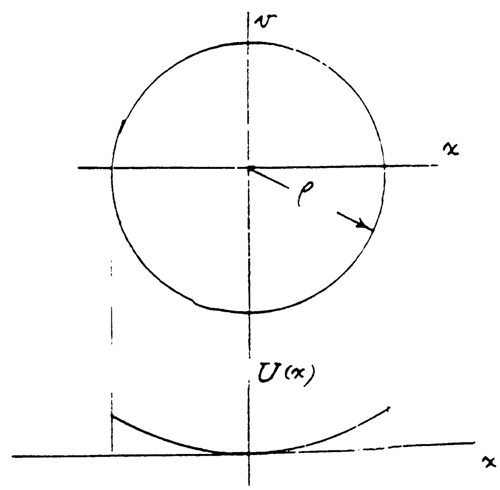
$\ddot{x} = \dot{x}\dfrac{d\dot{x}}{dx}$, Let $v = \dfrac{\dot{x}}{\omega_m}$

then above eq. becomes

$$\frac{dv}{dx} = -\frac{x}{v} \quad \text{or} \quad v^2 + x^2 = \rho^2$$

phase-plane traj. is a circle

$U = \frac{1}{2} k x^2$ a parabola



14-7

$$\tau = 4\int_0^{x_{max}} \frac{dx}{\sqrt{2[E - U(x)]}}$$

$U(x) = \frac{1}{2}\frac{k}{m}x^2$ per unit mass

$E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\frac{k}{m}x^2$

$\dot{x} = 0$ when $x = x_{max}$ $\therefore$ $x_{max} = \sqrt{\dfrac{2Em}{k}}$

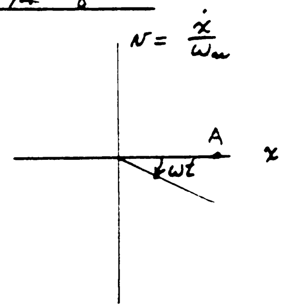
$$\tau = 4\int_0^{\sqrt{\frac{2Em}{k}}} \frac{dx}{\sqrt{2\left[E - \frac{1}{2}\frac{k}{m}x^2\right]}}$$

but $\frac{k}{m}x^2 = \omega_m^2 x^2$

$C^2 = 2E$

$$\tau = \frac{4}{\omega_m}\int_0^{u = \omega_m x_{max} = C} \frac{du}{\sqrt{C^2 - u^2}} = \frac{4}{\omega_m}\sin^{-1}\left(\frac{u}{C}\right)\Bigg|_0^C = \frac{4}{\omega_m}\frac{\pi}{2} = \frac{2\pi}{\omega_m}$$

14-8



$x(0) = A \qquad \dot{x}(0) = 0$

Let $y = \dot{x} \qquad \dot{y} = -\omega^2 x$

$$V = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{y^2 + \omega_m^4 x^2} = 0$$

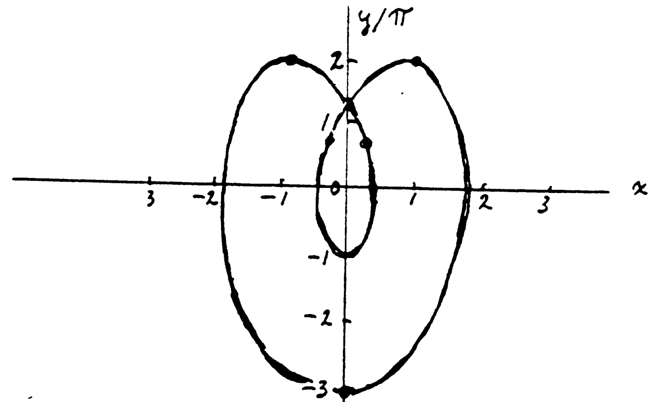
$V = 0$ only if $x = y = 0$

14-9

$$x = \cos \pi t + \sin 2\pi t$$

$$y = \dot{x} = -\pi \sin \pi t + 2\pi \cos 2\pi t$$

t	x	y
0	1	2π
.25	−.3	$.7\pi$
.50	0	$-\pi$
.75	.3	$.7\pi$
1.0	−1	2π
1.25	−1.7	$-.7\pi$
1.5	0	-3π
1.75	1.7	$-.7\pi$
2.0	1	2π



14-10

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

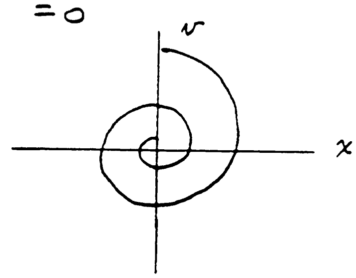
$$\dot{x}\frac{d\dot{x}}{dx} = -2\zeta\omega_n\dot{x} - \omega_n^2 x$$

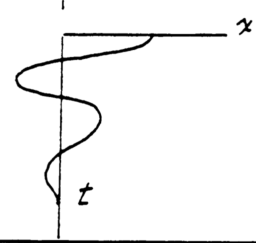
$$\frac{1}{\omega_n}\frac{d\dot{x}}{dx} = -2\zeta - \omega_n\frac{x}{\dot{x}}$$

Let $v = \dfrac{\dot{x}}{\omega_n}$

$$\frac{dv}{dx} = -2\zeta - \frac{x}{v}$$

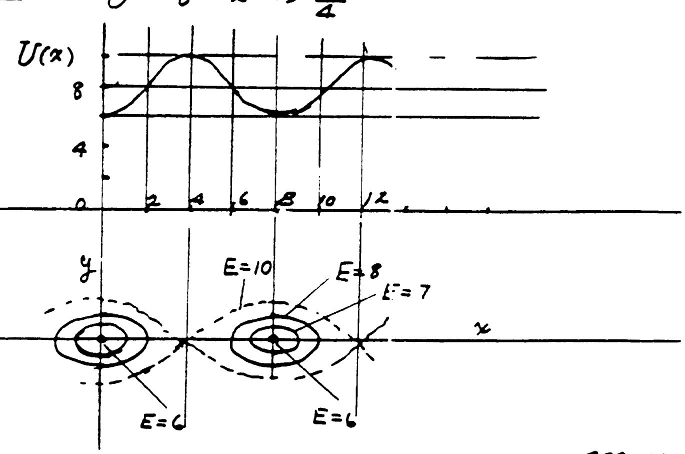
Trajectory is a spiral.





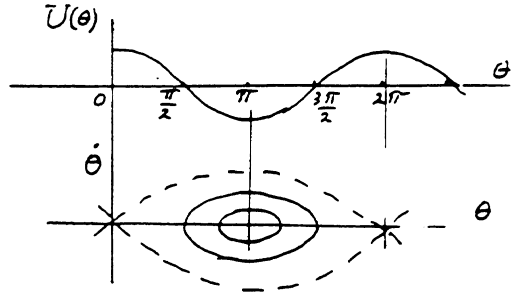
14-12

$$U = 8 - 2\cos\frac{\pi x}{4}$$



see next page for Prob 14-11

14-11 $\qquad U = \frac{g}{L}\cos\theta$



The origin of phase plane is shifted to π, as compared to Fig 14.4-2

$\theta = 0$ and 2π are unstable points

14-13 $\qquad \dfrac{dy}{dx} = \dfrac{2x + 2y}{5x - y} = \dfrac{P}{Q}$

singular points are $\dfrac{P}{Q} = \dfrac{0}{0}$

$$\begin{aligned} 2x + 2y &= 0 \\ 5x - y &= 0 \\ \hline 12x \quad &= 0 \end{aligned}$$

$\therefore\ x = 0, \quad y = 0$

$$\begin{Bmatrix} \dot{u} \\ \dot{v} \end{Bmatrix} = \begin{bmatrix} 5 & -1 \\ 2 & 2 \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \qquad \text{Eq. 14.3-8}$$

$(a+e) > 0 \quad \therefore$ system is unstable and aperiodic

$\lambda_1 = 3.0 \qquad u = e^{3t}$

$\lambda_2 = 4.0 \qquad v = e^{4t}$

14-14 $\qquad \begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} = \begin{bmatrix} 5 & -1 \\ 2 & 2 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} \qquad \begin{vmatrix} (5-\lambda) & -1 \\ 2 & (2-\lambda) \end{vmatrix} = 0$

gives $\lambda = \begin{cases} 3 \\ 4 \end{cases}$

subst λ into eq.

$(5-\lambda)x = y \quad$ gives $\quad x^{(1)} = .5\, y^{(1)}$, $\quad x^{(2)} = 1.0\, y^{(2)}$ $\qquad P = \begin{bmatrix} .5 & 1 \\ 1 & 1 \end{bmatrix}$

Transformation to decouple is $\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} .5 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \xi \\ \eta \end{Bmatrix}$

$$\begin{bmatrix} .5 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \dot{\xi} \\ \dot{\eta} \end{Bmatrix} = \begin{bmatrix} 5 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} .5 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \xi \\ \eta \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{\xi} \\ \dot{\eta} \end{Bmatrix} = \begin{bmatrix} .5 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 5 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} .5 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \xi \\ \eta \end{Bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{Bmatrix} \xi \\ \eta \end{Bmatrix}$$

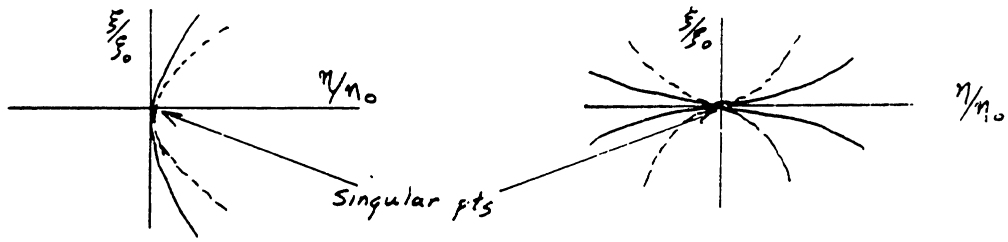
$\therefore\ \dot{\xi} = 3\xi \quad$ and $\quad \dot{\eta} = 4\eta \qquad$ decoupled eqs.

14-15 Uncoupled eqs. $\begin{cases} \dot{\xi} = \lambda_1 \xi \\ \dot{\eta} = \lambda_2 \eta \end{cases}$ can be written as

$$\frac{d\xi}{d\eta} = \left(\frac{\lambda_1}{\lambda_2}\right)\frac{\xi}{\eta} \qquad \text{Integrating} \quad \xi = \xi_0 \left(\frac{\eta}{\eta_0}\right)^{\frac{\lambda_1}{\lambda_2}}$$

For $\frac{\lambda_1}{\lambda_2} = .5$, the traj. are tangent to ξ as shown

If $\frac{\lambda_1}{\lambda_2} = 2$, the traj. are Tangent to η



14-16 $\frac{d\xi}{d\eta} = 2\frac{\xi}{\eta} \qquad \therefore \quad \xi = \xi_0 \left(\frac{\eta}{\eta_0}\right)^2$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} P \end{bmatrix} \begin{Bmatrix} \xi \\ \eta \end{Bmatrix} = \begin{bmatrix} P \end{bmatrix} \begin{Bmatrix} \xi_0 \left(\frac{\eta}{\eta_0}\right)^2 \\ \eta \end{Bmatrix}$$

$\therefore$ Eq for u & v are of the form

$$\left.\begin{aligned} u &= A\eta^2 + B\eta = x - x_s \\ v &= C\eta^2 + D\eta = y - y_s \end{aligned}\right\} \text{ only a linear shift of origin}$$

Need original eq.

$$\begin{Bmatrix} \dot{u} \\ \dot{v} \end{Bmatrix} = \begin{bmatrix} a & b \\ c & e \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \quad \text{corresponding to } \frac{\lambda_1}{\lambda_2} = 2$$

a, b, c, e must be known before plotting.

Use Eq. 12.3-9

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \begin{Bmatrix} \xi \\ \eta \end{Bmatrix} = P \begin{Bmatrix} \xi \\ \eta \end{Bmatrix}$$

From Eq. 14.3-8

$$\begin{vmatrix} (a-\lambda) & b \\ c & (e-\lambda) \end{vmatrix} = 0 \quad \therefore \quad \frac{u_1}{v_1} = \frac{-b}{a-\lambda_1}, \quad \frac{u_2}{v_2} = \frac{-b}{a-\lambda_2}$$

14-16 Cont. Since only relative values of u, v are essential, Let $v_1 = v_2 = 1.0$ and $u_1 = \frac{-b}{a-\lambda_1}$, $u_2 = \frac{-b}{a-\lambda_2}$

Using Eq. 14.3-8 we have

$$P\begin{Bmatrix}\dot{\xi}\\ \dot{\eta}\end{Bmatrix} = \begin{bmatrix}a & b\\ c & e\end{bmatrix}P\begin{Bmatrix}\xi\\ \eta\end{Bmatrix}$$

$$\begin{Bmatrix}\dot{\xi}\\ \dot{\eta}\end{Bmatrix} = P^{-1}\begin{bmatrix}a & b\\ c & e\end{bmatrix}P\begin{Bmatrix}\xi\\ \eta\end{Bmatrix} = [\Lambda]\begin{Bmatrix}\xi\\ \eta\end{Bmatrix}$$

$$\therefore P^{-1}\begin{bmatrix}a & b\\ c & e\end{bmatrix}P = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix} \text{ where } P = \begin{bmatrix}\frac{-b}{a-\lambda_1} & \frac{-b}{a-\lambda_2}\\ 1 & 1\end{bmatrix}$$

Equating the two elements on the two sides of the above eq,

$$[(au_2+b)-(cu_2+e)] = 0$$

$$[-u_2(au_1+b)+u_1(cu_1+e)] = 0$$

$$\frac{1}{(u_1v_2-u_2v_1)}[(au_1+b)-(cu_1+e)] = \lambda_1$$

$$\frac{1}{(u_1v_2-u_2v_1)}[-u_2(au_2+b)+u_1(cu_2+e)] = \lambda_2$$

These 4 eqs. can be solved for a, b, c, e, then subst. into Eq. 14.3-9

Other alternative is to solve for u + v from Eq. 14.3-12 for chosen values of t.

14-17 From given eq.

$$\begin{Bmatrix}\dot{v}\\ \dot{u}\end{Bmatrix} = \begin{bmatrix}\alpha & \beta\\ -\beta & \alpha\end{bmatrix}\begin{Bmatrix}v\\ u\end{Bmatrix} \qquad \begin{vmatrix}(\alpha-\lambda) & -\beta\\ -\beta & (\alpha-\lambda)\end{vmatrix} = 0$$

$$\therefore \lambda^2 - 2\alpha\lambda + \alpha^2 + \beta^2 = 0 \quad \text{and} \quad \lambda = -\alpha \pm i\beta$$

14-18 $u = \rho\cos\theta$ $\qquad v = \rho\sin\theta$

$du = d\rho\cos\theta - \rho\sin\theta\, d\theta \qquad dv = d\rho\sin\theta + \rho\cos\theta\, d\theta$

subst. u + v in Eq. for Prob. 14-17

$$\frac{dv}{du} = \frac{\beta\cancel{\rho}\cos\theta + \alpha\cancel{\rho}\sin\theta}{\alpha\cancel{\rho}\cos\theta - \beta\cancel{\rho}\sin\theta} = \frac{d\rho\sin\theta + \rho\cos\theta\, d\theta}{d\rho\cos\theta - \rho\sin\theta\, d\theta}$$

$$\left(\frac{d\rho}{\rho}\sin\theta + \cos\theta\, d\theta\right)(\alpha\cos\theta - \beta\sin\theta) = \left(\frac{d\rho}{\rho}\cos\theta - \sin\theta\, d\theta\right)(\beta\cos\theta + \alpha\sin\theta)$$

$$\frac{d\rho}{\rho}\left[\alpha\cancel{\cos\theta\sin\theta} - \beta(\sin^2\theta + \cos^2\theta) - \alpha\cancel{\sin\theta\cos\theta}\right] = -d\theta\left[\alpha\cos^2\theta - \cancel{\beta\sin\theta\cos\theta} + \cancel{\beta\sin\theta\cos\theta} + \alpha\sin^2\theta\right]$$

$$\therefore \frac{d\rho}{\rho} = \frac{\alpha}{\beta}d\theta \qquad \rho = e^{\frac{\alpha}{\beta}\theta}$$

14-19 Eq. in x, y plane is $xy = \pm C$

node point at origin is unstable

$$x\,dy + y\,dx = 0 \quad \therefore \quad \frac{dy}{dx} = -\frac{y}{x}$$

$$\begin{Bmatrix} \dot{y} \\ \dot{x} \end{Bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} y \\ x \end{Bmatrix} \qquad \begin{vmatrix} -(1+\lambda) & 0 \\ 0 & (1-\lambda) \end{vmatrix} = 0$$

$$(1-\lambda)(1+\lambda) = 0 \qquad \lambda = \pm 1 \qquad \xi = \xi_0 e^{t}, \quad \eta = \eta_0 e^{-t}$$

$$\frac{d\xi}{d\eta} = \frac{\lambda_1}{\lambda_2}\frac{\xi}{\eta} = -\frac{\xi}{\eta} \quad \text{or} \quad \eta\, d\xi + \xi\, d\eta = 0$$

$\therefore$ ξ, η plot is $\xi\eta = \pm$ const. $\therefore$ same

---

14-20 The form of the eq. is

$$\frac{dv}{du} = \frac{v+u}{u} \qquad \begin{Bmatrix} \dot{v} \\ \dot{u} \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} v \\ u \end{Bmatrix}$$

characteristic eq.

$$\begin{vmatrix} (1-\lambda) & 1 \\ 0 & (1-\lambda) \end{vmatrix} = 0$$ Leads to two equal roots $\lambda = 1$

Transformation eq. 14.3-9 cannot be applied

14-21 Differentiate the eq. $x^2 + 2xy + 3y^2 = C$

$$2x\,dx + 2x\,dy + 2y\,dx + 6y\,dy = 0$$

$$(2x + 6y)\,dy = -(2x + 2y)\,dx \quad \therefore \quad \frac{dy}{dx} = \frac{-x-y}{x+3y}$$

14-22 Let $v = \dot{x}$ then the eq. $\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = 0$

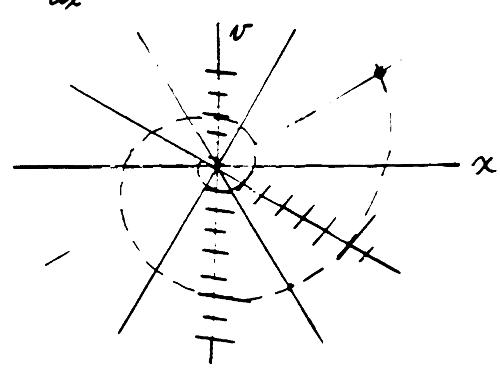
becomes $\dot{v} + 2\zeta\omega_n v + \omega_n^2 x = 0$, $\quad v\dfrac{dv}{dx} + 2\zeta\omega_n v + \omega_n^2 x = 0$

Let $\dfrac{dv}{dx}$ = constant for isocline $= C_1$

$v(C_1 + 2\zeta\omega_n) + \omega_n^2 x = 0$

= straight line through origin

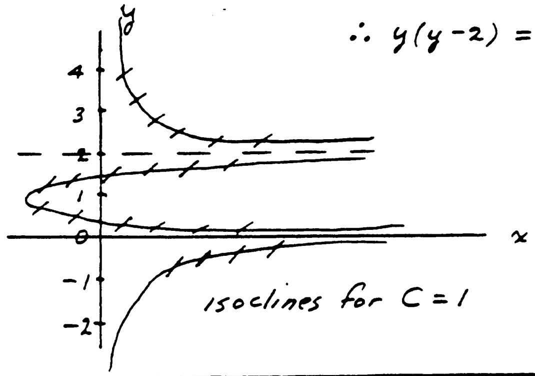
As $t \to \infty$ the points on the isoclines move towards the origin $\therefore$ stable



14-23 $\quad \frac{dy}{dx} = xy(y-2) =$ constant for isocline

$$\therefore \; y(y-2) = \frac{C}{x}$$

y	$y(y-2)$	x for $C=1$
-3	15	.066
-2	8	.125
-1	3	.333
0	0	∞
1	-1	-1
2	0	∞
3	3	.333
4	8	.125



isoclines for $C = 1$

14-24 $$y\frac{dy}{dx} + \omega_m^2 x + \mu x^3 = 0$$

integrating $$y^2 + \omega_m^2 x^2 + \tfrac{1}{2}\mu x^4 = 2E$$

or $$2T + 2U = 2E$$

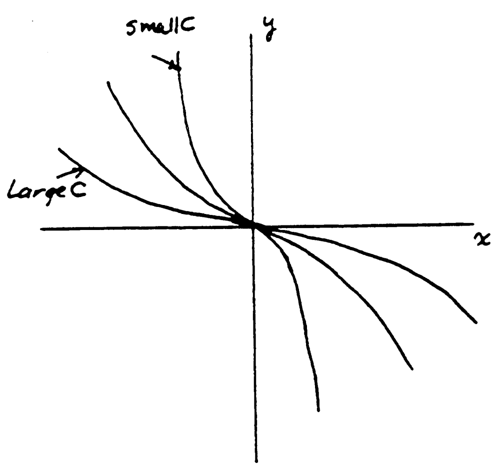
Since $y = \frac{dx}{dt}$ $\quad dt = \frac{dx}{y} = \frac{dx}{\sqrt{2E - 2U}}$

For $\frac{1}{4}$ period $$\frac{\tau}{4} = \int_0^A \frac{dx}{\sqrt{2(E-U)}}$$

14-25 $\quad$ From Prob. 14-24 $\quad \frac{dy}{dx} = \frac{-\omega_m^2 x - \mu x^3}{y} = C$

Let $\frac{\omega_m^2}{C} = 4 + \frac{\mu}{C} = 2$

x	$4+2x^2$	$y=-x(4+2x^2)$
-1	6	6
0	4	0
1	6	-6
2	12	-24
3	22	-66

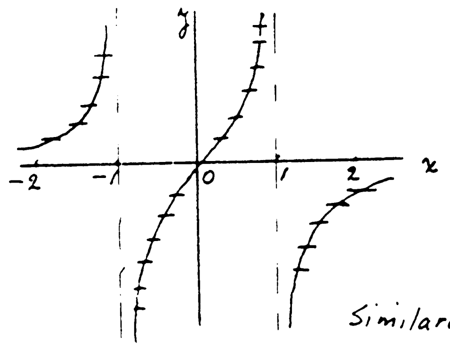


14-26 $\quad \ddot{x} - \mu \dot{x}(1-x^2) + x = 0$

$$y\frac{dy}{dx} - \mu y(1-x^2) + x = 0$$

$$\frac{dy}{dx} = \frac{\mu y(1-x^2) - x}{y} = \mu(1-x^2) - \frac{x}{y} = \text{const}$$

For $\frac{dy}{dx} = C = 0$ and $\mu = 2$



x	y
0	0
$\pm .2$	$\pm .104$
$\pm .4$	$\pm .239$
$\pm .6$	$\pm .45$
$\pm .8$	± 1.11
$\pm .9$	± 2.37
± 1.0	$\pm \infty$
± 2	$\mp .333$
± 4	$\mp .13$

Similarly solve for $C = \pm 1$ and other values

14-27 $\quad$ Let $\omega_n^2 = \frac{k}{m} \qquad \tau = \omega_n t \qquad y = \frac{dx}{d\tau}$

Then $\dot{x} = \omega_n y \qquad \ddot{x} = \omega_n^2 y \frac{dy}{dx}$

the eq $\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x + \frac{\mu}{m}x^3 = 0$ becomes

$$y\frac{dy}{dx} + \frac{c}{m}y + x + \frac{\mu}{\omega_n^2 m}x^3 = 0$$

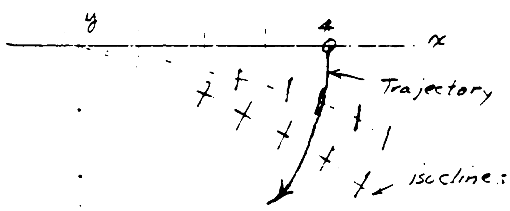
$$\frac{dy}{dx} = \frac{-\left[x + \left(\frac{c}{m}y + \frac{\mu}{\omega_n^2 m}x^3\right)\right]}{y} = \alpha$$

14-28 $\quad$ Above eq. can be written as $\qquad$ given: $\omega_n^2 = 25$, $\frac{\mu}{m} = 5$

$(\alpha + \frac{c}{m})y = -(x + \frac{\mu}{\omega_n^2 m}x^3)$ $\quad$ where $\alpha = \frac{dy}{dx}$ $\qquad \frac{c}{m} = 2.0$

$\therefore (\alpha + 2)y = -(x + 0.2x^3)$ $\quad$ Assign different values for α and plot the field of isoclines. From prob. 14-27 $\dot{x}(0) = \omega_n y(0) = 0$

$\therefore$ Start with $x(0) = 4$, $\dot{x}(0) = 0 \therefore y(0) = 0$ and fill in trajectory



All isoclines are vertical along x axis. The right side of eq. is straight line + cubic which must be $\div$ by $(\alpha + 2)$

14 – 29 $\ddot{\theta} + \frac{g}{l}\sin\theta = 0$ Let $x = \theta$, $\omega_0^2 = g/l$

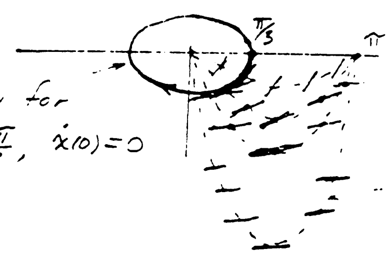
$$y = \frac{\dot{\theta}}{\omega_0} = \frac{\dot{x}}{\omega_0} = \frac{dx}{d\tau}$$

$\therefore\ \omega_0^2\, y \frac{dy}{dx} + \frac{g}{l}\sin x = 0$ with $\alpha = \frac{dy}{dx}$ = constant

$\alpha y = -\left(\frac{g}{l\omega_0^2}\right)\sin x$ or $y = -\frac{g}{\alpha l \omega_0^2}\sin x$ = eq. for isocline with α = const.

Since $\frac{g}{l\omega_0^2} = 1$

$y = -\frac{1}{\alpha}\sin x$ = isocline



Trajectory for $\theta(0) = x(0) = \frac{\pi}{3}$, $\dot{x}(0) = 0$

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14 – 30 $\ddot{\theta} + \frac{g}{l}\sin\theta = 0$ $\dot{\theta}\frac{d\dot{\theta}}{d\theta} + \frac{g}{l}\sin\theta = 0$

Integrate $\frac{\dot{\theta}^2}{2} - \frac{g}{l}\cos\theta = E$ $\therefore\ U = -\frac{g}{l}\cos\theta$

At $t=0$ $\theta = 60^\circ$ $\dot{\theta} = 0$ $\therefore$ $0 - \frac{g}{l}\cos 60^\circ = E$

Since $\dot{\theta} = \frac{d\theta}{dt}$ $dt = \frac{d\theta}{\dot{\theta}}$ $E = -\frac{g}{2l} = -\frac{g}{l}\cos\theta_0$

$$dt = \frac{d\theta}{\sqrt{2E + 2\frac{g}{l}\cos\theta}} \qquad t = \int \frac{d\theta}{\sqrt{2\left(E + \frac{g}{l}\cos\theta\right)}}$$

$$t = \sqrt{\frac{l}{g}}\int \frac{d\theta}{\sqrt{2(\cos\theta - \cos\theta_0)}} \qquad \cos\theta = 1 - 2\sin^2\frac{\theta}{2}$$

$$\cos\theta - \cos\theta_0 = 2\left(\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right)$$

Let $\sin\frac{\theta}{2} = \sin\frac{\theta_0}{2}\sin\phi = k\sin\phi$ (i)

then $\cos\theta - \cos\theta_0 = 2\sin^2\frac{\theta_0}{2}\left(1 - \sin^2\phi\right) = 2\sin^2\frac{\theta_0}{2}\cos^2\phi$ (ii)

Diff (i) $\frac{1}{2}\cos\frac{\theta}{2}\,d\theta = k\cos\phi\,d\phi$ $\therefore$ $d\theta = \frac{2k\cos\phi\,d\phi}{\cos\frac{\theta}{2}}$ (iii)

Subst (ii) & (iii) into t

$$t = \sqrt{\frac{l}{g}}\int \frac{2k\cos\phi\,d\phi}{\cos\frac{\theta}{2}\ 2\sin\frac{\theta_0}{2}\cos\phi} = \sqrt{\frac{l}{g}}\int \frac{d\phi}{\cos\frac{\theta}{2}}$$

14-30 Cont:

$$t = \sqrt{\frac{l}{g}} \int \frac{d\phi}{\sqrt{1-\sin^2\frac{\theta}{2}}} = \sqrt{\frac{l}{g}} \int \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$$

$\therefore$ period $\tau = 4\sqrt{\frac{l}{g}} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$ $\qquad k = \sin\frac{\theta_0}{2}$

Since $\sin\phi = \frac{\sin\frac{\theta}{2}}{\sin\frac{\theta_0}{2}}$, when $\theta = 0$, $\phi = 0$

when $\theta = \theta_0$, $\phi = \frac{\pi}{2}$ $\quad = \frac{1}{4}$ cycle

---

14-31 $\quad \ddot{x} + \omega_n^2 x + C\,\mathrm{sgn}(\dot{x}) = 0 \qquad \tau = \omega_n t$

$\dot{x}\frac{d\dot{x}}{dx} + \omega_n^2 x + C\,\mathrm{sgn}(\dot{x}) = 0 \qquad$ Let $\frac{dx}{d\tau} = y = \frac{\dot{x}}{\omega_n}$

$\omega_n^2 y \frac{dy}{dx} + \omega_n^2 x + C\,\mathrm{sgn}(y) = 0$

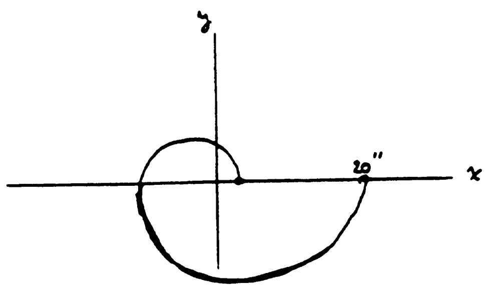
$$\frac{dy}{dx} = -\frac{\frac{1}{\omega_n}C\,\mathrm{sgn}(y) + x}{y} = -\frac{f(y)+x}{y}$$

where $f(y) = \frac{1}{\omega_n}C\,\mathrm{sgn}(y)$

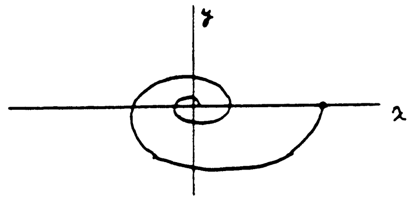
14-32 $\quad$ Initial values $\quad x(0) = 20^\circ \qquad y(0) = 0$

$\omega_n = \sqrt{\frac{3.60}{.10}} = 6, \qquad \mu = 0.20$

$\frac{\mu g}{\omega_n^2} = \frac{.20 \times 386}{36} = 2.145$ in



14-33 For the undamped pendulum the trajectory is an ellipse, For the damped pendulum the curve is inside of the ellipse, as shown



14-34
$$\ddot{\theta} + \omega_m^2 \sin\theta = 0 \qquad \sin\theta \cong \theta - \frac{\theta^3}{6}$$

$$\ddot{\theta} + \omega_m^2\left(\theta - \frac{\theta^3}{6}\right) = 0$$

Let $\theta = \theta_0 + \mu\theta_1$ and $\omega^2 = \omega_m^2 + \mu\alpha$

From Eq. 12.6-9

$$\omega^2 = \omega_m^2 + \frac{3}{4}\mu A^2 = \omega_m^2\left[1 + \frac{3}{4}\times\frac{1}{6}\theta_0^2\right]$$

$$\therefore \quad \omega = \omega_m\sqrt{1 + \frac{1}{8}\theta_0^2} \cong \omega_m\left(1 + \frac{1}{16}\theta_0^2\right) \qquad \omega_m = \sqrt{\frac{g}{\ell}}$$

14-35 From Prob 14-34

$$\omega = \frac{2\pi}{\tau} \cong \frac{2\pi}{\tau_m}\left(1 + \frac{1}{16}\theta_0^2\right) \quad \therefore \tau \cong \tau_m\left(\frac{1}{1 + \frac{1}{16}\theta_0^2}\right)$$

14-36
$$\ddot{x} + 0.15\dot{x} + 10x + x^3 = 5\cos(\omega t + \varphi)$$

$$\omega_m^2 = 10, \qquad c = .15, \qquad \mu = 1.0, \qquad F = 5$$

Eq. 14.6-11 becomes

$$25 = \left[(10 - \omega^2)A + \frac{3}{4}A^3\right]^2 + \left[.15\,\omega A\right]^2$$

Rearrange to

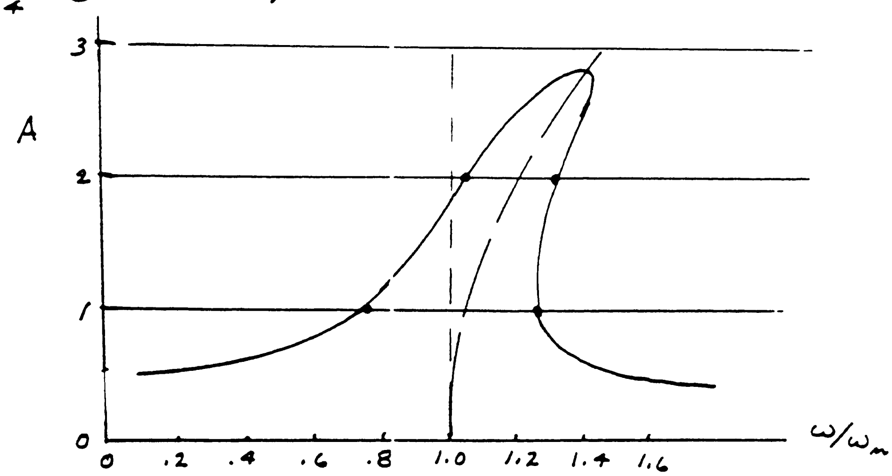
$$\omega^4 - \left(20 + \frac{3}{2}A^2 - .0225\right)\omega^2 + \left(100 + 15A^2 + \frac{9}{16}A^4 - \frac{25}{A^2}\right) = 0$$

$$\omega^4 \quad - \quad b\,\omega^2 \quad + \quad c \quad = 0$$

14-36 Cont.

A	b	c	ω^2	ω/ω_m
0	19.98	$-\infty$	∞	∞
1	21.48	90.5	$\begin{cases} 5.7 \\ 15.78 \end{cases}$	$\begin{cases} .755 \\ 1.255 \end{cases}$
2	25.98	162.7	$\begin{cases} 10.69 \\ 15.29 \end{cases}$	$\begin{cases} 1.036 \\ 1.24 \end{cases}$
3	32.48	272.7	complex ←	this indicates the peak to be below A = 3
$\cong$ .5				0

$\sqrt{\frac{b^2}{4} - c} = 0$ gives $A \cong 2.9$



14-37

ω/ω_m vs φ_1 & φ_2

φ_1 = branch 1

φ_2 = " 2

$$\tan\phi = \frac{B_0}{A_0} = \frac{c\omega}{(\omega_n^2 - \omega^2) + \frac{3}{4}\mu A^2} \quad \text{Eq. 14.6-10}$$

$$= \frac{.15\omega}{(10 - \omega^2) + \frac{3}{4}A^2}$$

subst. ω & A from Prob. 14-36

$$(\tan\phi)_{A=1} = \frac{.15\sqrt{5.7}}{(10 - 5.7) + .75} = \frac{.358}{5.05} = .071, \quad \phi = 4^\circ\,4'$$

14-37 Cont:

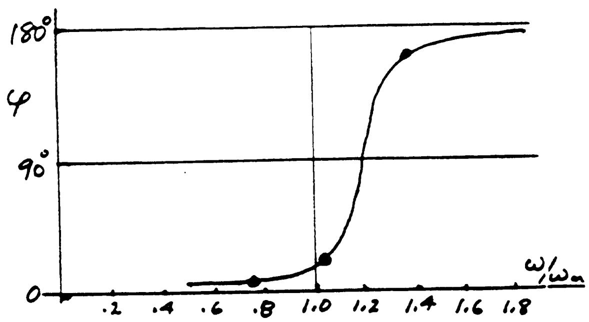
$$(\tan\phi)_{A=1} = \frac{.15\sqrt{15.78}}{(10-15.78)+.75} = \frac{.60}{-5.03} = -.119 \qquad \phi = 173^\circ 12'$$

$$(\tan\phi)_{A=2} = \frac{.15\sqrt{10.69}}{(10-10.69)+3} = \frac{.490}{2.31} = .213 \qquad \phi = 12^\circ$$

$$= \frac{.15\sqrt{15.29}}{(10-15.29)+3} = \frac{.590}{-2.29} = -.257 \qquad \phi = 185^\circ 35'$$

$$(\tan\phi)_{A=0} = \frac{\infty}{-\infty^2} = -\frac{1}{\infty} = -0 \qquad \phi = 180^\circ -$$

$$(\tan\phi)_{A\equiv .5} = \frac{0}{(10-0)+.375} = 0 \qquad \phi = 0^\circ$$



ω/ω_n	φ_1	φ_2
0	0	
.755	4°4′	
1.036	12°	
1.25		173° 12′
1.24		185° 35′
∞		180°

14-38 Moment eq. about an accelerating point A is

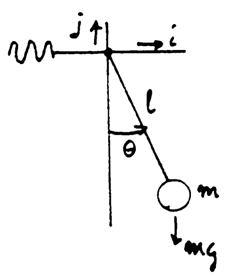
$$\vec{M}_A = I_A\,\dot{\vec{\omega}} + \vec{\rho}_{AC} \times m\,\vec{a}_A$$ (see Dynamics by Pestel & Thomson p 213 McGraw Hill.)

where c is the center of mass and $\vec{\rho}_{AC}$ is a vector from A to C. For this problem $y_A = y_0 \cos 2\omega t$

$I_A = m\ell^2 \qquad |\rho_{AC}| = \ell \qquad |a_A| = -4y_0\omega^2\cos 2\omega t$

$$\vec{\rho}_{AC} \times m\vec{a}_A = \ell(\sin\theta\,\vec{i} - \cos\theta\,\vec{j}) \times m(-4y_0\omega^2\cos 2\omega t)\,\vec{j}$$

$$= -4y_0\omega^2\ell\, m\cos 2\omega t \cdot \vec{k}$$



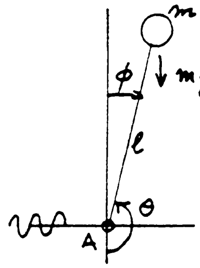
$$\dot{\vec{\omega}} = \ddot{\theta}\,\vec{k} \qquad \vec{M}_A = -mg\ell\sin\theta\,\vec{k}$$

$$\therefore\ -mg\ell\sin\theta = m\ell^2\ddot{\theta} - 4y_0\omega^2\ell m\cos 2\omega t$$

$$\ddot{\theta} + \left(\frac{g}{\ell} - \frac{4y_0\omega^2\cos 2\omega t}{\ell}\right)\sin\theta = 0$$

14-39 For the inverted pendulum $\theta = \pi - \phi$

$\sin\theta = \sin\phi$



$$\therefore\ -\ddot{\phi} + \left(\frac{g}{\ell} - \frac{4\omega_0^2 y_0}{\ell}\cos 2\omega t\right)\sin\phi = 0$$

For small ϕ

$$\ddot{\phi} + \left(-\frac{g}{\ell} + \frac{4\omega_0^2 y_0}{\ell}\cos 2\omega t\right)\phi = 0$$

Compare with Eq. (g) Sec. 1.... which is

$$\frac{d^2y}{dz^2} + (a - 2b\cos 2z)\,y = 0$$

which results in

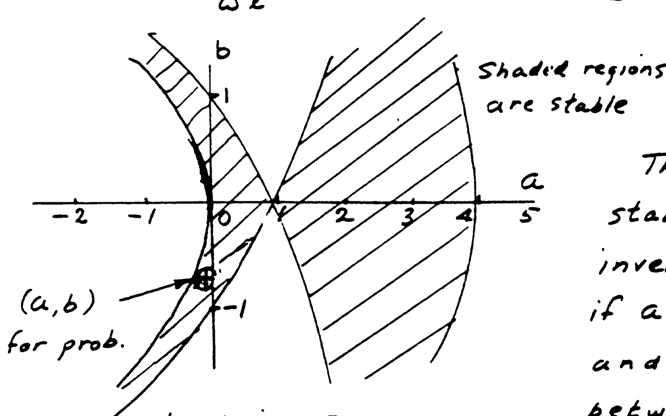
$$z = \omega t, \qquad y = \phi, \qquad dz^2 = \omega^2 dt$$

Rewrite eq.

$$\omega^2\frac{d^2\phi}{dz^2} + \left(-\frac{g}{\ell} + \frac{4\omega^2 y_0}{\ell}\cos 2z\right)\phi = 0$$

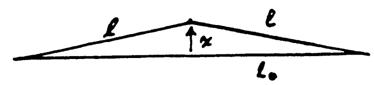
$$\frac{d^2\phi}{dz^2} + \left(-\frac{g}{\omega^2\ell} + \frac{4y_0}{\ell}\cos 2z\right)\phi = 0$$

$$\therefore\ a = -\frac{g}{\omega^2\ell} \qquad \text{and} \qquad b = -\frac{2y_0}{\ell}$$



Mathieu Plot.

The plot indicates that stable oscillations of the inverted pendulum is possible if a is a small number and b is also negative between 0 & -1, as shown by the point ⊕.

14-40 New length after displ is 

$\ell = \ell_0 (1 + \frac{x^2}{\ell_0^2})^{1/2} \cong \ell_0 (1 + \frac{1}{2} \frac{x^2}{\ell_0^2})$

Let T_0 = initial tension. Since increase in length is $\frac{x^2}{\ell_0}$, the increase in tension is $K \frac{x^2}{\ell_0}$. Total tension $= (T_0 + K \frac{x^2}{\ell_0})$

Eq. of motion

$$m\ddot{x} = -\frac{x}{\ell} 2 \left(T_0 + K \frac{x^2}{\ell_0} \right) \qquad \frac{x}{\ell} \cong \frac{x}{\ell_0}$$

$$m\ddot{x} + \left(\frac{2T_0}{\ell_0} \right) x + \left(\frac{2K}{\ell_0^2} \right) x^3 = 0$$

Assume sol. $x = x_1 + \mu x_2 + \cdots$ where μ = arbitrary parameter and $x_2 \ll x_1$. Then $x^3 \cong x_1^3 + 3\mu x_1^2 x_2$. Subst. in D.E.

$$m\ddot{x}_1 + \frac{2T_0}{\ell_0} x_1 + \alpha x_1^3 = 0 \qquad \alpha = \frac{2K}{\ell_0^2}$$

$$\mu \left[m\ddot{x}_2 + \frac{2T_0}{\ell_0} x_2 + 3\alpha x_1^2 x_2 \right] = 0$$

If α is small, then $x_1 = A \cos \omega_n t$, $\omega_n = \sqrt{\frac{2T_0}{\ell_0}}$ and 2^{nd} eq. becomes

$$m\ddot{x}_2 + \frac{2T_0}{\ell_0} x_2 + (3\alpha A^2 \cos^2 \omega_n t) x_2 = 0$$

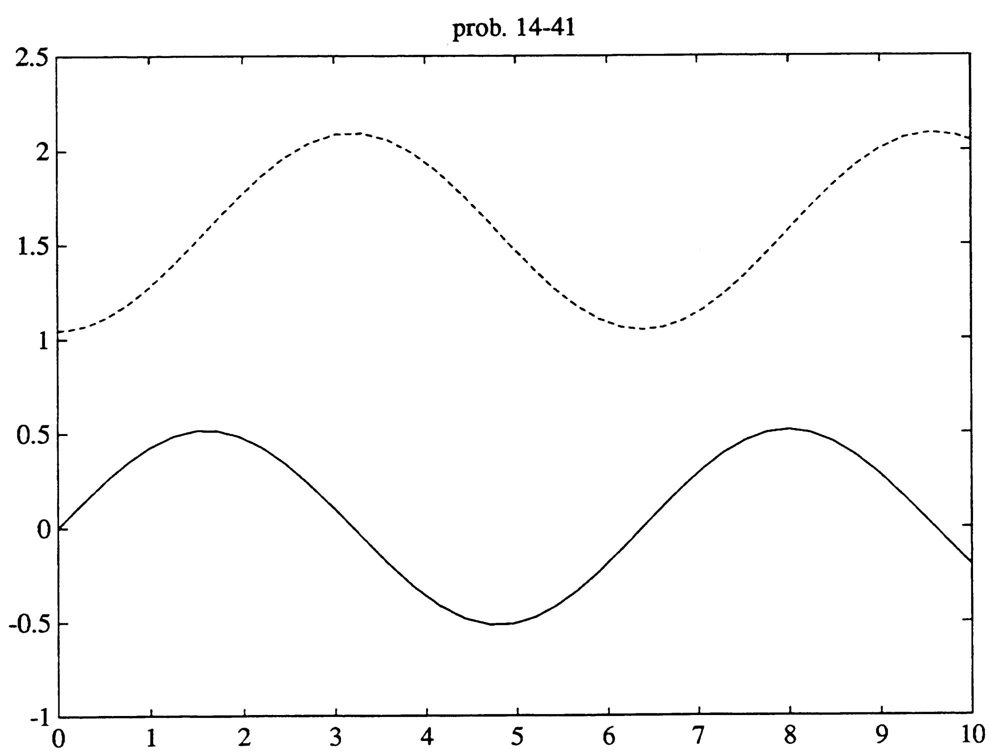
$$m\ddot{x}_2 + \left[\left(\frac{2T_0}{\ell_0} + \frac{3}{2}\alpha A^2 \right) + \frac{3}{2}\alpha A^2 \cos 2\omega_n t \right] x_2 = 0$$

which is Mathieu eq.

14-41

```
»t0=0

t0 =

     0

»tf=10

tf =

    10

»x0=[0, pi/3]

x0 =

         0    1.0472

»[t,x]=ode23('prob1441',t0,tf,x0);
»plot(t,x)
```

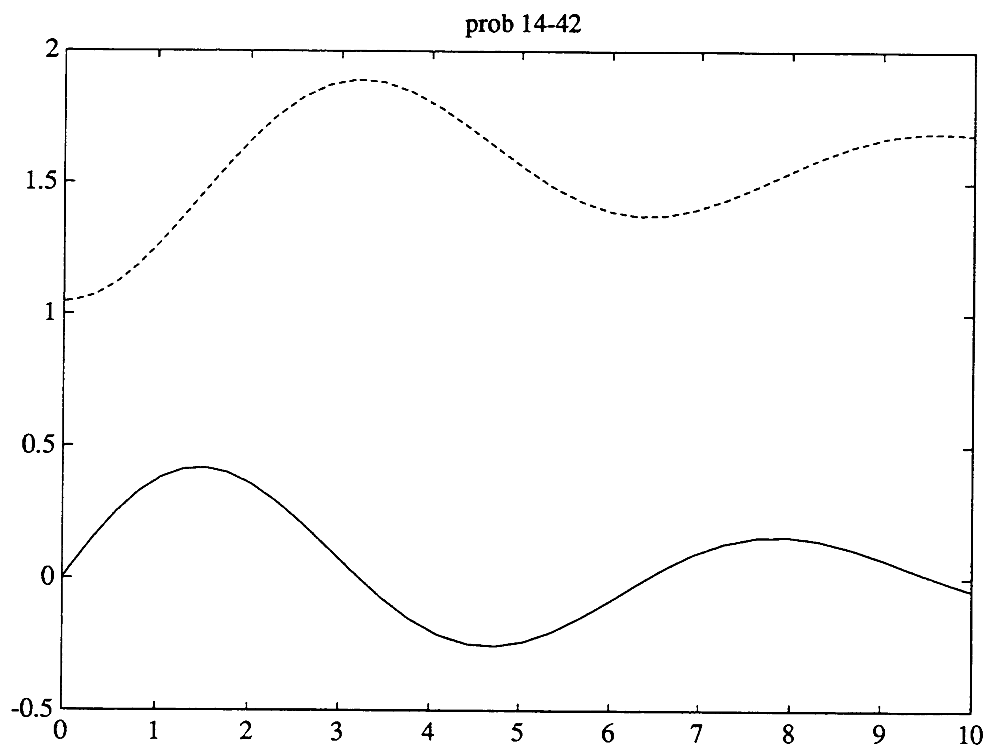
```
function xdot=prob1441(t,x)
xdot=[sin(x(2));x(1)];
```



14-42

Input the intial condition as in problem 14-41

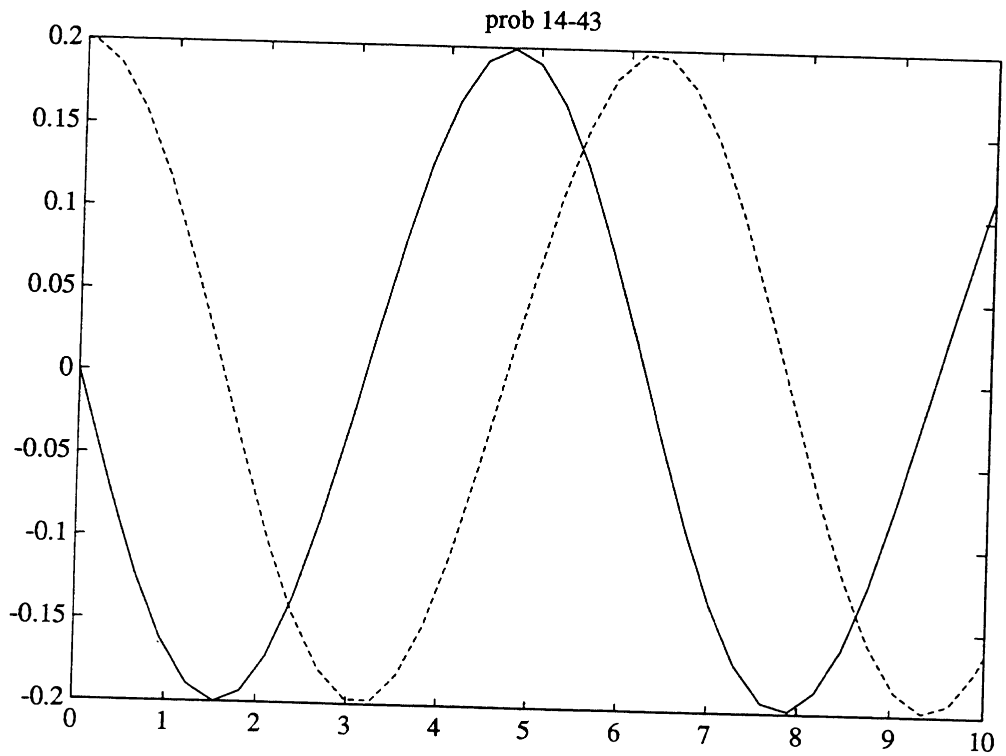
```
function xdot=prob1442(t,x)
xdot=[sin(x(2))-.3*x(1);x(1)];
```



14-43

Input The initial conditions as in problem 14-41

```
function xdot=prob1443(t,x)
xdot=[-x(2)-.2*x(2)^3;x(1)];
```



14-44

```
clear
global m g l F0 w
'Problem 14-44 Solution'
m=input('Enter the value of the mass (kg) : ')
l=input('Enter the value of the pendulum length (m) : ')
F0=input('Enter the value of the forcing amplitude (N) : ')
w=input('Enter the value of the forcing frequency (rad/s) : ')
x0=input('Enter the two dimensional initial condition [x xd] : ')
t0=input('Enter the value of the initial time (s) : ')
Tf=input('Enter the value of the final time (s) : ')
g=9.81;
xp0=[x0  w*t0];
T0=0;
[T,xp]=ode45('p1444d',T0,Tf,xp0);
m=size(xp);n=m(1,1);
j=1;
y1(1)=xp(1,1);
y2(1)=xp(1,2);
plot(y1(1),y2(1),'.')
hold on
for i=2:n
     xpo(3)=2*pi*j+xp0(3);
     if xp(i,3)-xpo(3)>0 & xp(i-1,3)-xpo(3)<0
    j=j+1;
       s1=(xp(i,1)-xp(i-1,1))/(xp(i,3)-xp(i-1,3));
      y1(j)=s1*(xpo(3)-xp(i-1,3))+xp(i-1,1);
       s2=(xp(i,2)-xp(i-1,2))/(xp(i,3)-xp(i-1,3));
      y2(j)=s2*(xpo(3)-xp(i-1,3))+xp(i-1,2);
     plot(y1(j),y2(j),'.')
   hold on
   end
    if xp(i,3)==xpo(3)
    j=j+1;
     y1(j)=xp(i,1);
     y2(j)=xp(i,2);
      plot(y1(j),y2(j),'.')
   hold on
   end
end
xlabel('theta');ylabel('thetadot');title('Problem  14-44:Poincare  Section')
```

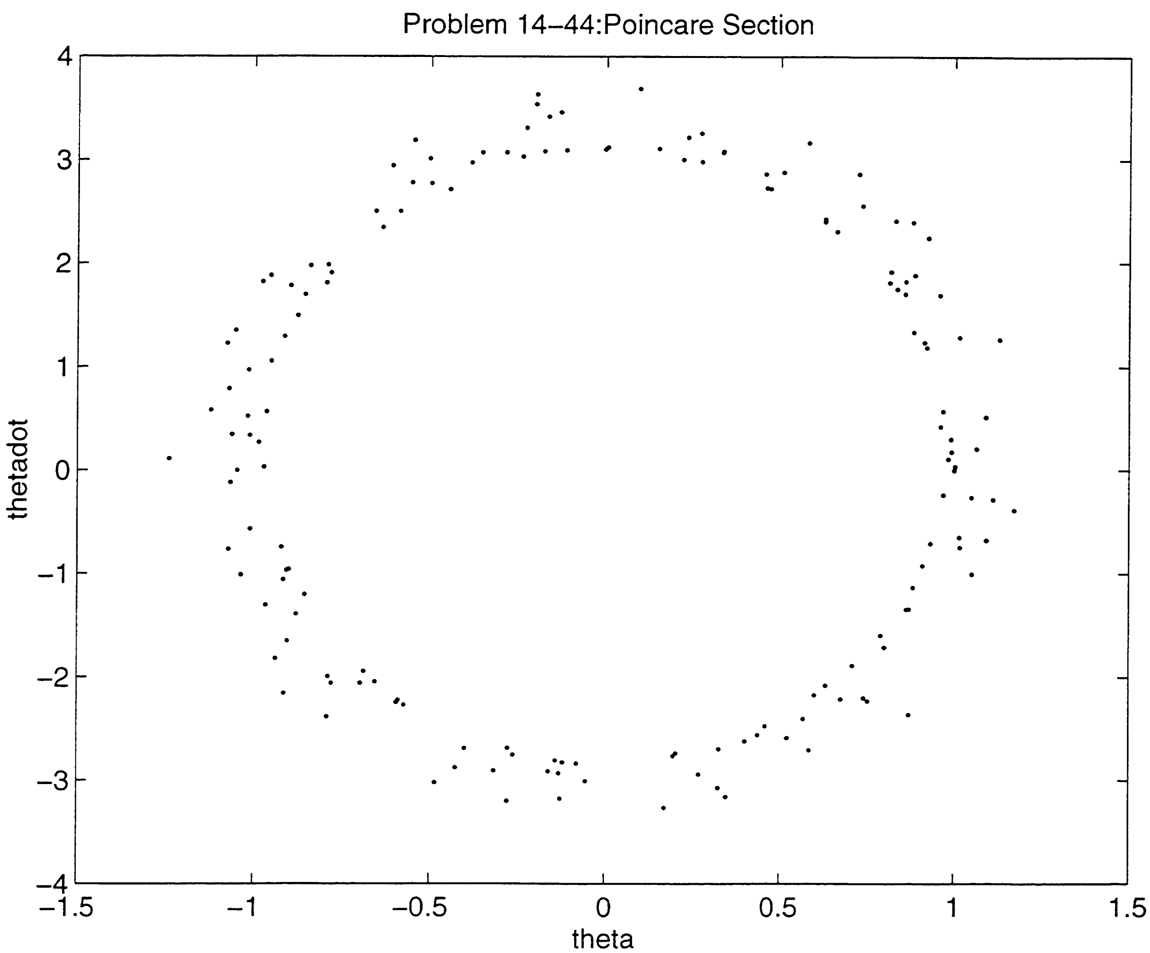
14-45

```
clear
global m c k mu F w
'Problem 14-45 Solution'
m=input('Enter the value of the mass (kg) : ')
c=input('Enter the value of the damping coefficient (N.m/s) : ')
k=input('Enter the value of the spring stiffness (N/m): ')
mu=input('Enter the value of mu (N/m^3) with its sign : ')
F=input('Enter the value of the forcing amplitude (N) : ')
w=input('Enter the value of the forcing frequency (rad/s) : ')
x0=input('Enter the two dimensional initial condition [x xd] : ')
t0=input('Enter the value of the initial time (s) : ')
Tf=input('Enter the value of the final time (s) : ')
xp0=[x0  w*t0];
T0=0;
[T,xp]=ode45('p1445d',T0,Tf,xp0);
m=size(xp);n=m(1,1);
j=1;
y1(1)=xp(1,1);
y2(1)=xp(1,2);
plot(y1(1),y2(1),'.')
hold on
for i=2:n
    xpo(3)=2*pi*j+xp0(3);
    if xp(i,3)-xpo(3)>0 & xp(i-1,3)-xpo(3)<0
   j=j+1;
      s1=(xp(i,1)-xp(i-1,1))/(xp(i,3)-xp(i-1,3));
     y1(j)=s1*(xpo(3)-xp(i-1,3))+xp(i-1,1);
      s2=(xp(i,2)-xp(i-1,2))/(xp(i,3)-xp(i-1,3));
     y2(j)=s2*(xpo(3)-xp(i-1,3))+xp(i-1,2);
    plot(y1(j),y2(j),'.')
  hold on
  end
   if xp(i,3)==xpo(3)
   j=j+1;
    y1(j)=xp(i,1);
    y2(j)=xp(i,2);
     plot(y1(j),y2(j),'.')
  hold on
  end
end
xlabel('x');ylabel('xdot');title('Problem  14-45:Poincare  Section')
```

14-45 cont

```
function  xdot=p1444d(t,x)
global m g l F0 w
xdot=[x(2);
        -(g/l)*sin(x(1))+(F0/(m*l))*sin(x(3));
    w];
```

$m = l = F0 = w = 1, \quad x_0 = [1 \quad 0], \quad t_0 = 0, \quad T_f = 1000$



14-45 cont

```
function  xdot=p1445d(t,x)
global m c k mu F w
xdot=[x(2);
        (1/m)*(-c*x(2)-k*x(1)-mu*x(1)^3+F*cos(x(3)));
    w];
```

$x_0 = [0 \quad 0]$, $t_0 = 0$, $T_f = 1000$